Nucleation and Propagation of Fracture and Healing in Elastomers: A Phase-Transition Theory & Numerical Implementation

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Classical experiments: Gent-Park test

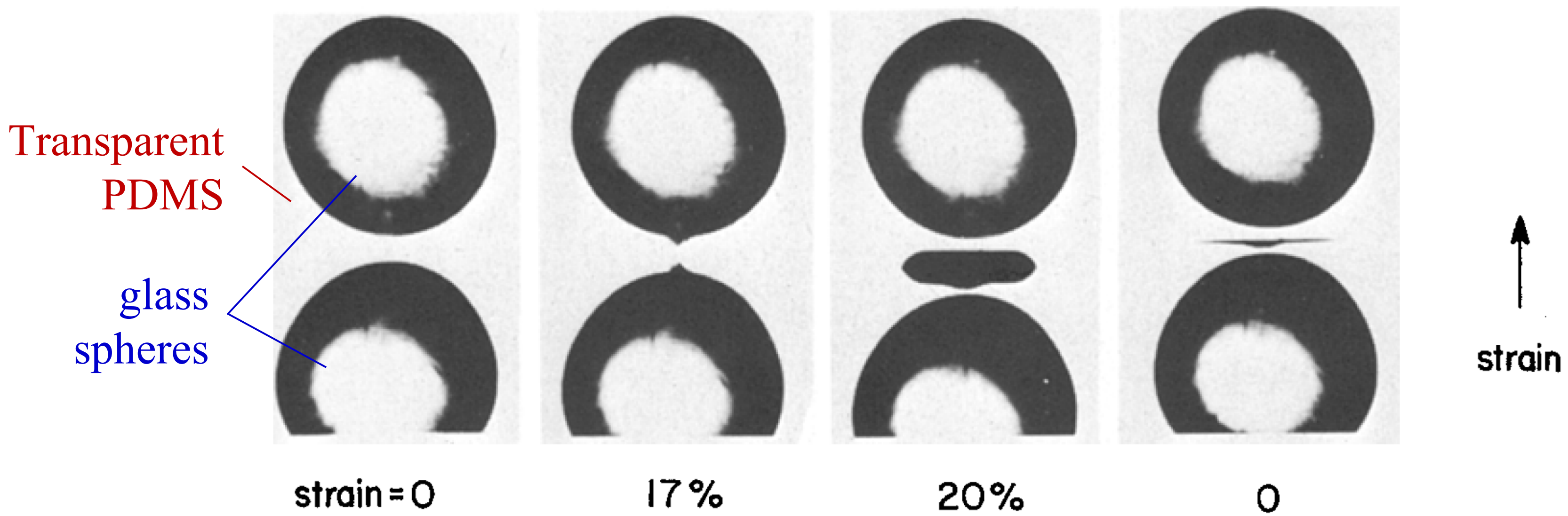
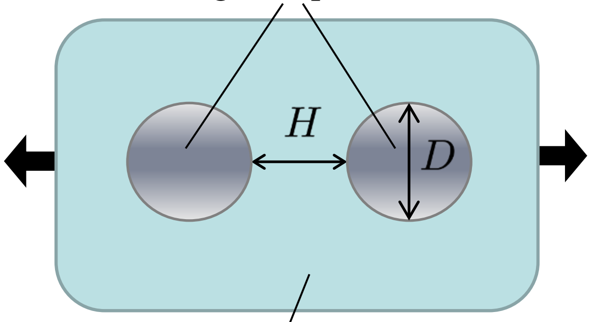


Figure 14 Progress of cavitation in a silicone elastomer, $E = 2.2$ MPa, containing two glass beads of 1.25 mm diameter bonded to the elastomer. Direction of applied tension: vertical.

Gent and Park (1984)

The Gent-Park experiment revisited



- **Geometry**

$$\left.\begin{aligned} D &= 2.300\,\mathrm{mm} \\ H &= 0.483\,\mathrm{mm} \end{aligned}\right\} \quad \frac{H}{D} = 0.21$$

- **Rate of macroscopic loading**

$\sim 5\mathrm{mm/min}$

- **Spatiotemporal resolution**

$1\,\mathrm{pixel} \approx 1\,\mu\mathrm{m}$

$15\,\mathrm{frames/s}$

Poulain, Lefèvre, Lopez-Pamies, Ravi-Chandar *Int. J. Frac.* (2017)

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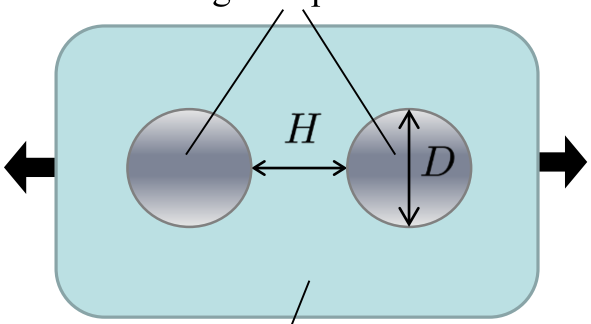

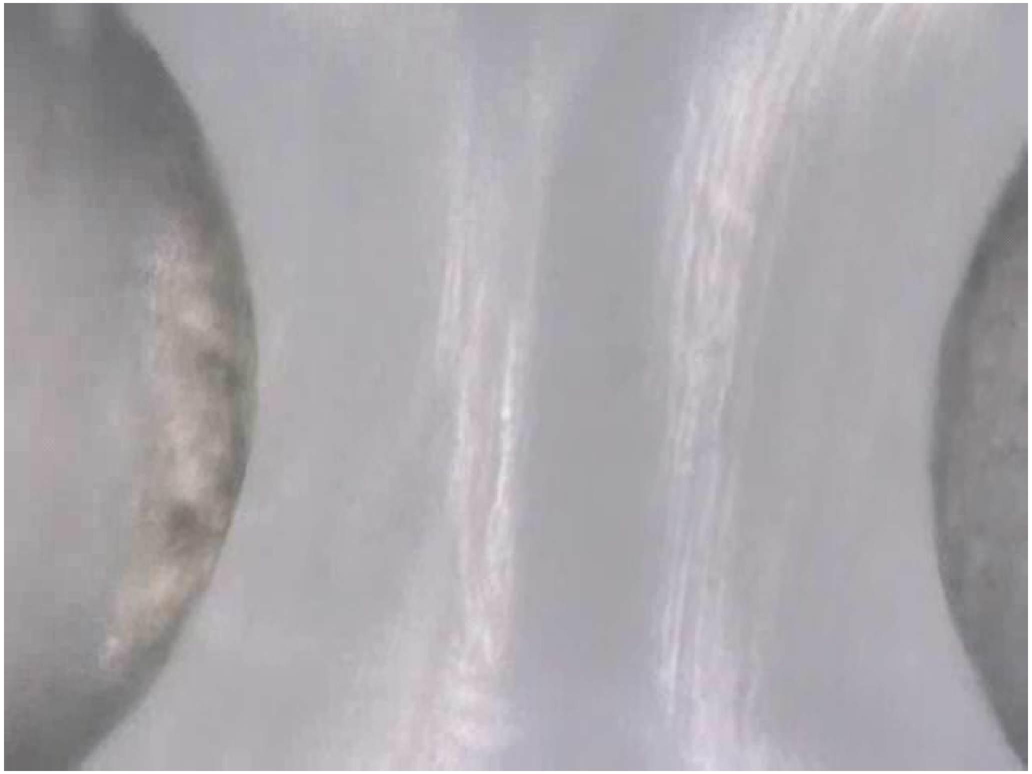

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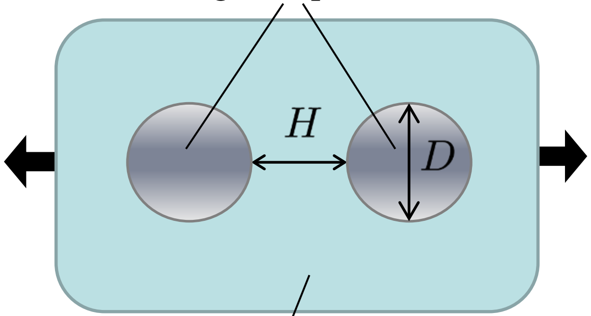

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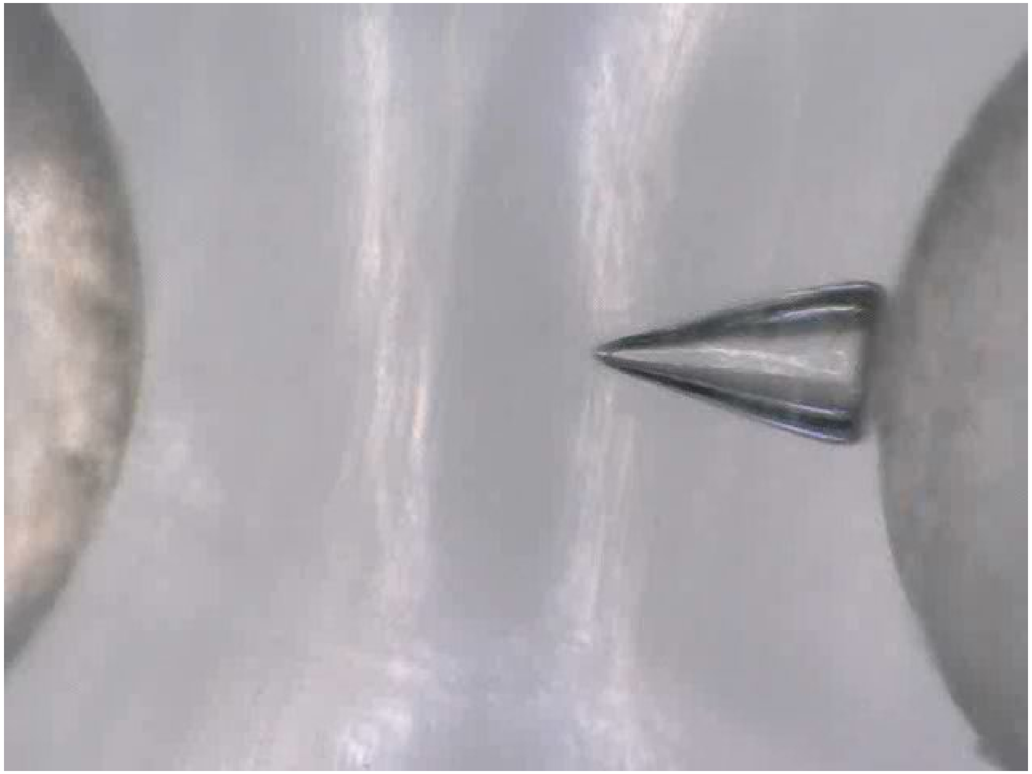

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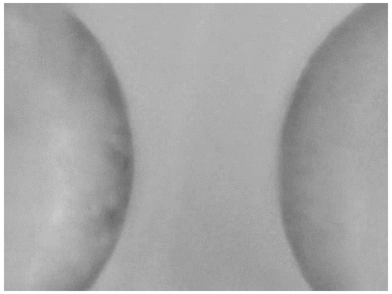
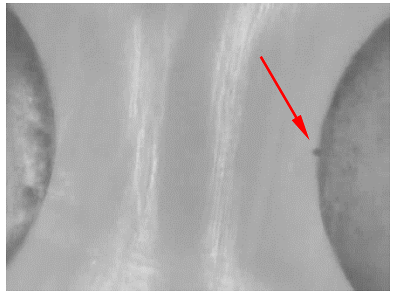
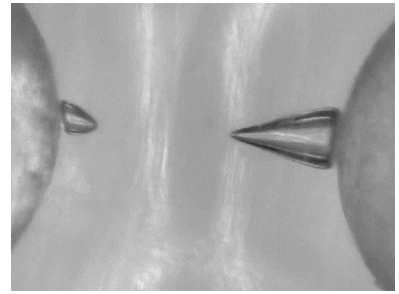


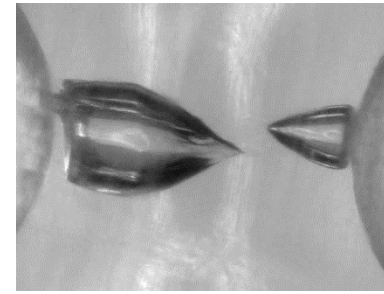
(4000) $\lambda = 1.62$



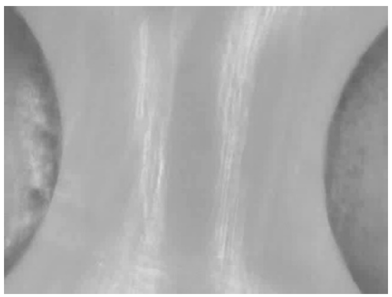
(7575) $\lambda = 2.95$



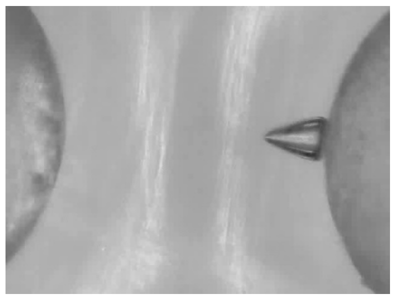
(14100) $\lambda = 3.10$



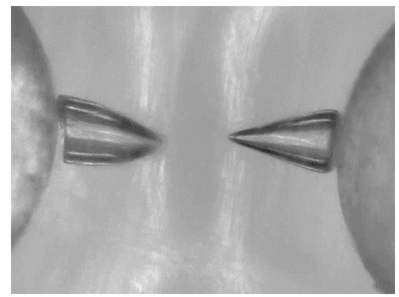
(14850) $\lambda = 3.39$



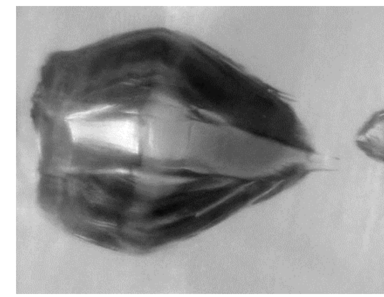
(7574) $\lambda = 2.95$



(7582) $\lambda = 2.95$

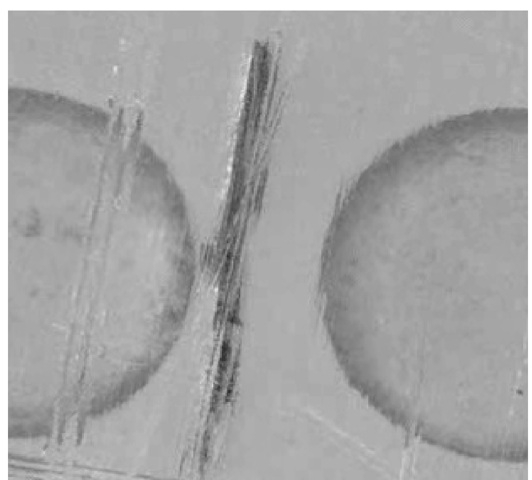


(14750) $\lambda = 3.15$

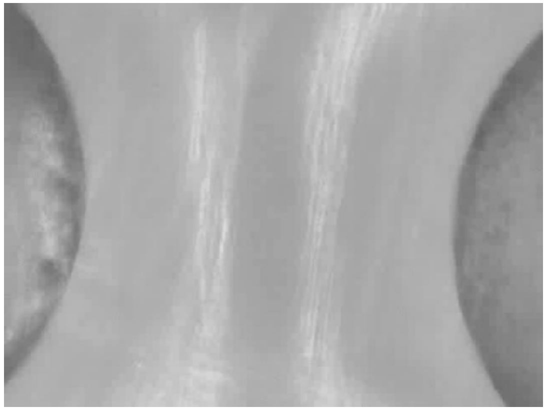


(15000) $\lambda = 4.28$

unloaded



The Gent-Park experiment revisited



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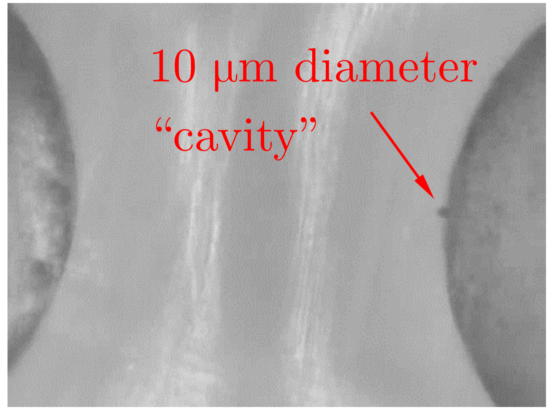
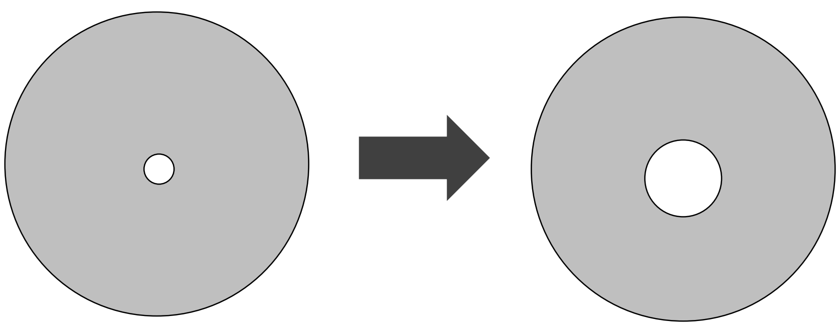


$(7575)\ \lambda = 2.95$

Remark I: The nucleation of “cavities” is stochastic in nature

Remark II: The nucleation of “cavities” is an extremely fast process involving stretch rates in excess of **100 s^{-1}**

Remark III: The nucleation of “cavities” is fundamentally a by-product of fracture and not solely of elasticity



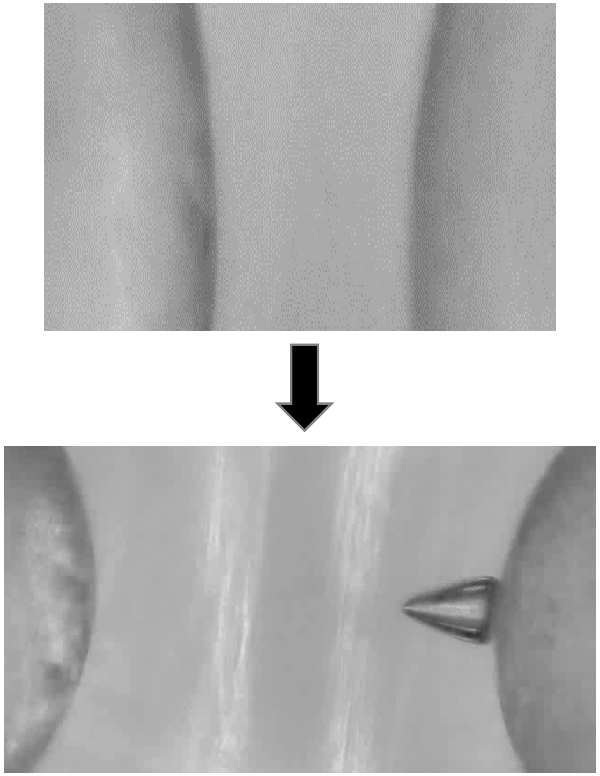
δ diameter submicron defect

10 μm diameter “cavity”

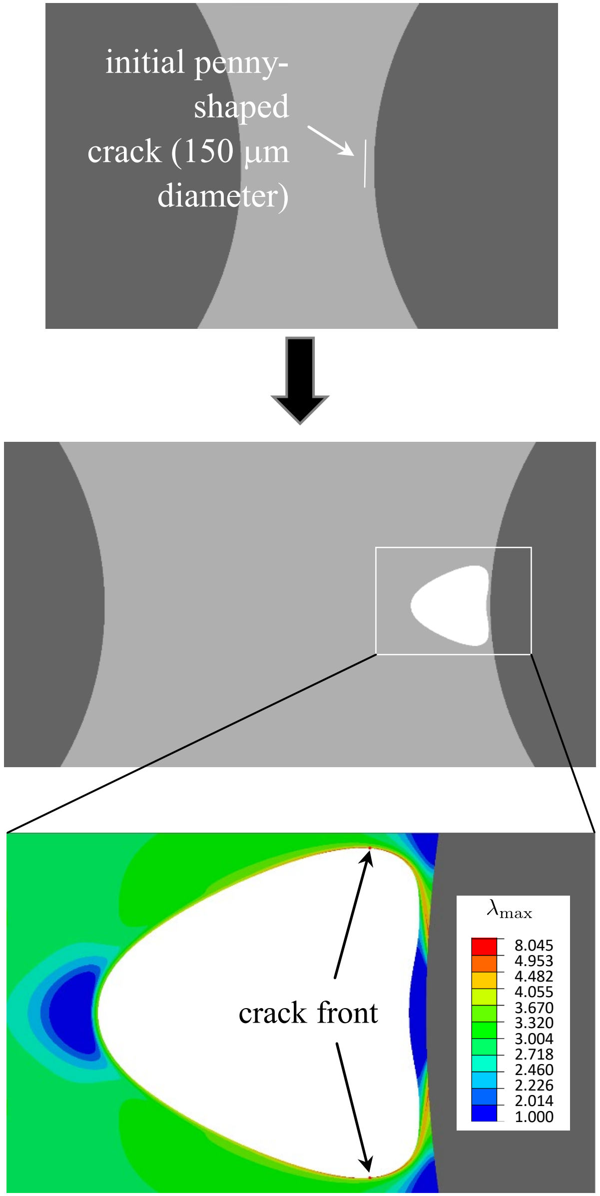
$$\begin{cases} \delta = 100\,\mathrm{nm} \rightarrow \lambda_{cavity} = 100 \\ \delta = 10\,\mathrm{nm} \rightarrow \lambda_{cavity} = 1000 \end{cases}$$

A crack or a cavity?

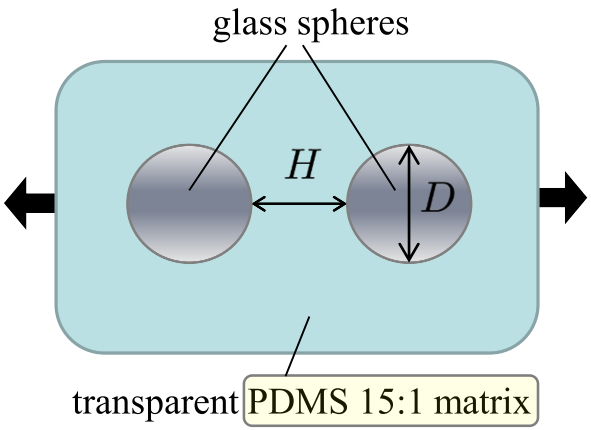
Experiment



Simulation



The Gent-Park experiment revisited



- **Geometry**

$$\left.\begin{aligned} D &= 3.178\,\mathrm{mm} \\ H &= 0.340\,\mathrm{mm} \end{aligned}\right\} \quad \boxed{\frac{H}{D} = 0.107}$$

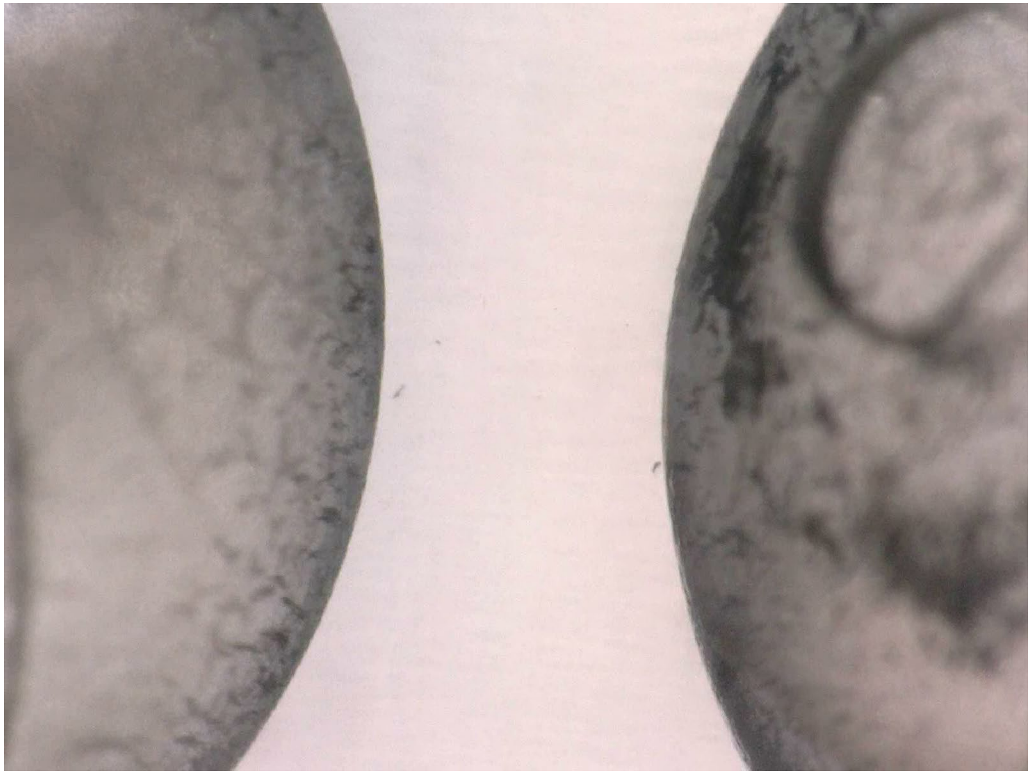
- **Rate of macroscopic loading**

$\sim 5\mathrm{mm/min}$

- **Spatiotemporal resolution**

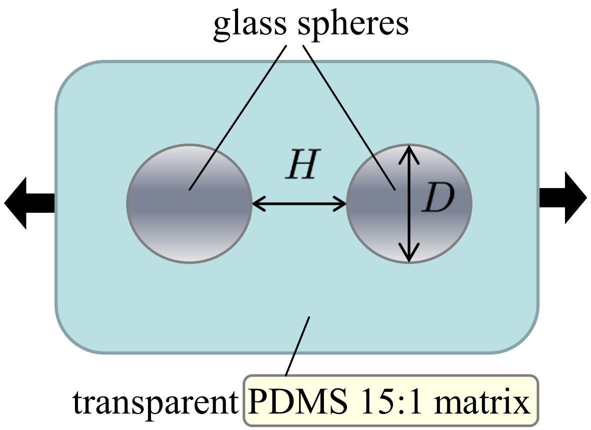
$1\,\mathrm{pixel} \approx 1\,\mu\mathrm{m}$

$15\,\mathrm{frames/s}$



Poulain, Lefèvre, Lopez-Pamies, Ravi-Chandar *Int. J. Frac.* (2017)

The Gent-Park experiment revisited



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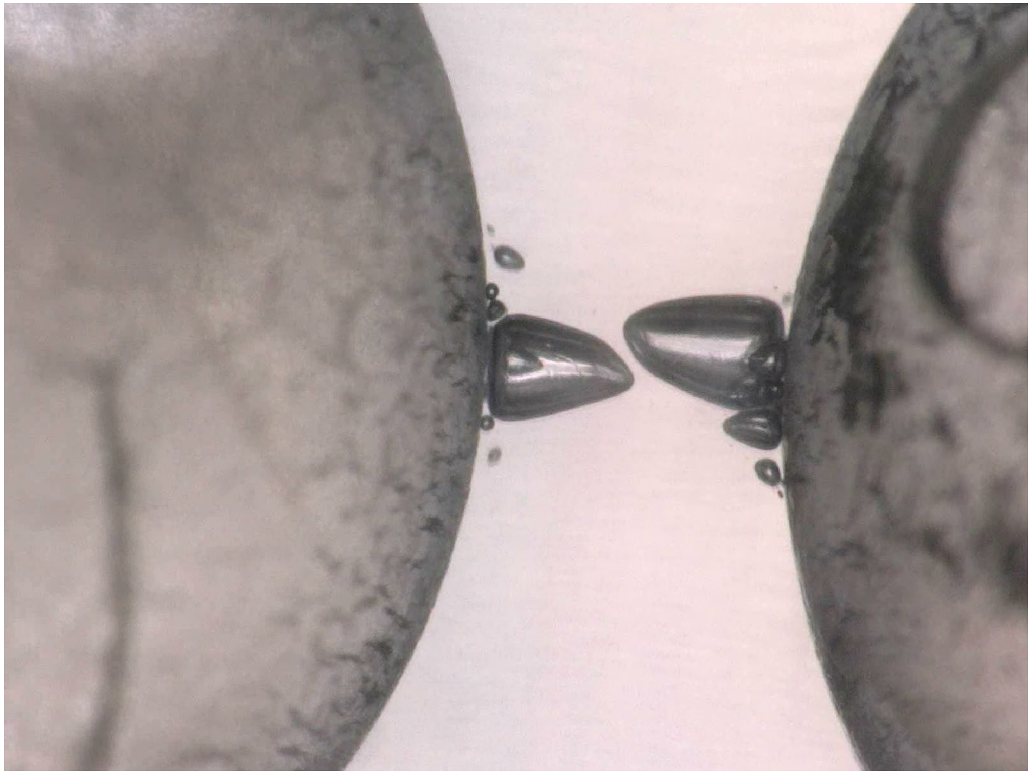

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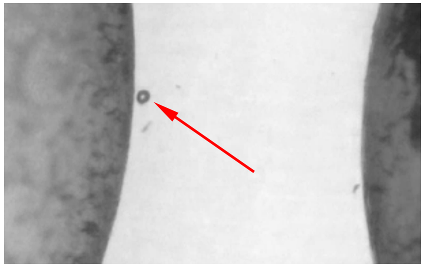
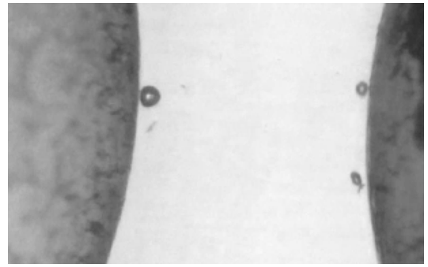
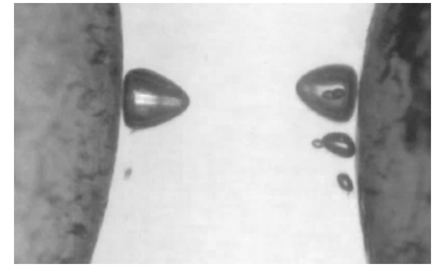
The Gent-Park experiment revisited



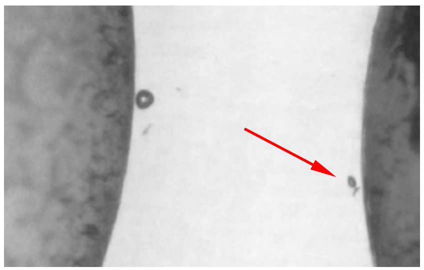
(440) $\lambda = 1.701$



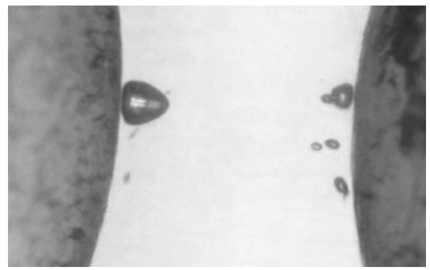
(461) $\lambda = 1.700$



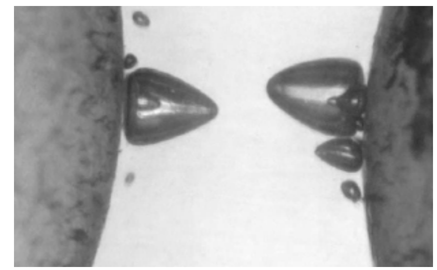
(735) $\lambda = 1.750$



(451) $\lambda = 1.714$

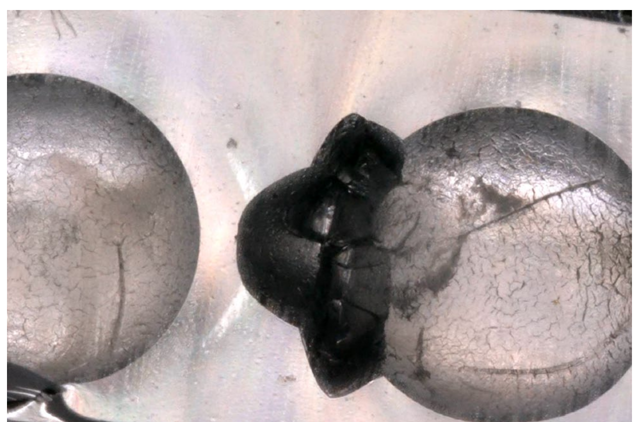


(585) $\lambda = 1.731$

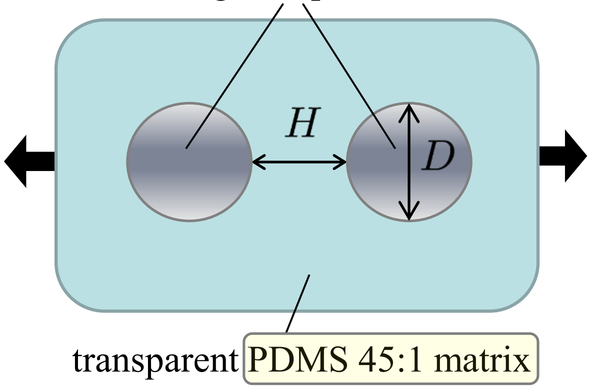


(1035) $\lambda = 1.778$

debonding



The Gent-Park experiment revisited



- **Geometry**

$$\left.\begin{aligned} D &= 2.543\,\mathrm{mm} \\ H &= 0.193\,\mathrm{mm} \end{aligned}\right\} \boxed{\frac{H}{D} = 0.076}$$

- **Rate of macroscopic loading**

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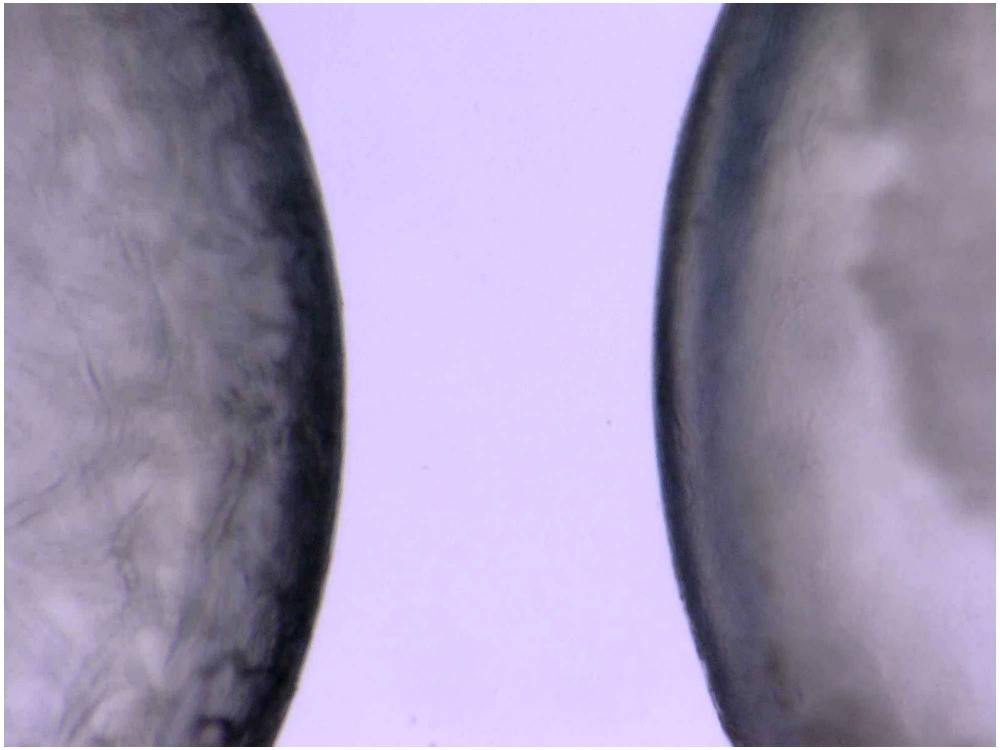
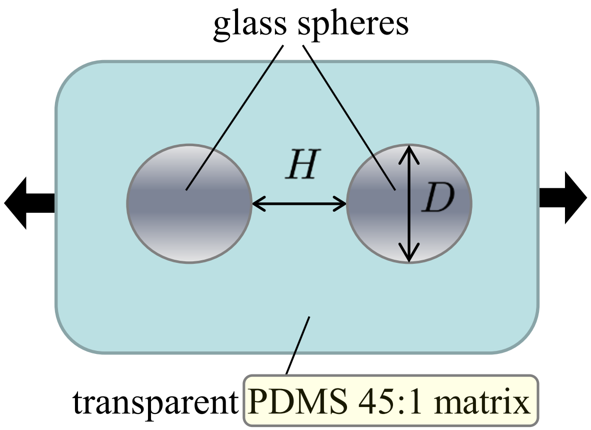
1st cycle



Poulain, Lopez-Pamies, Ravi-Chandar *Soft Matter* (2018)

The Gent-Park experiment revisited



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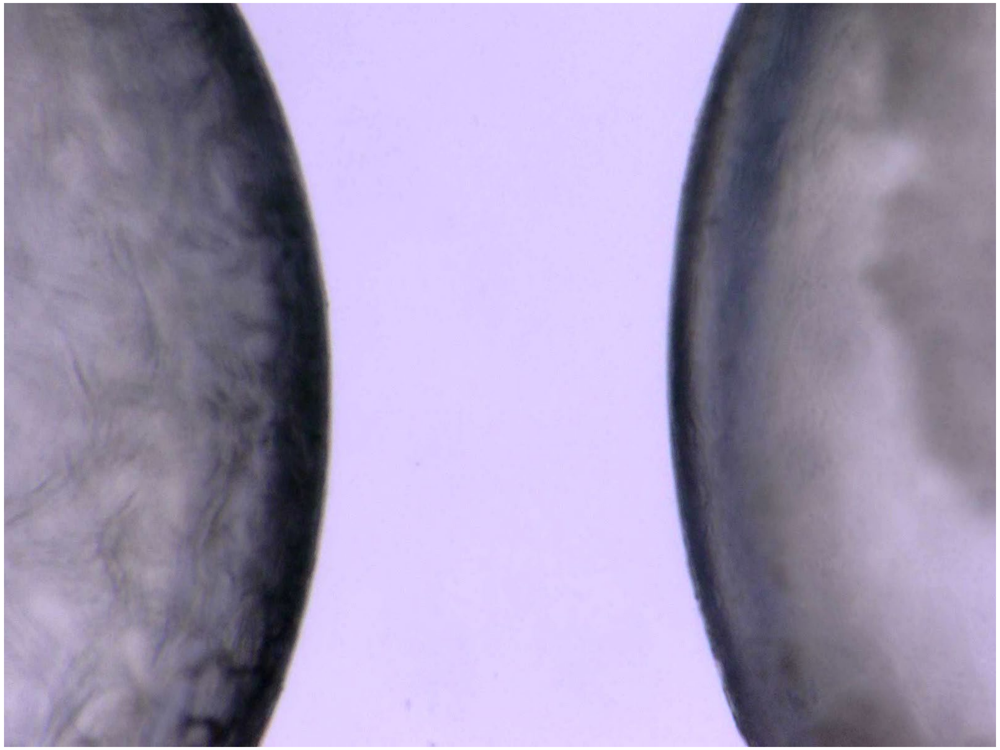
2^{nd} cycle



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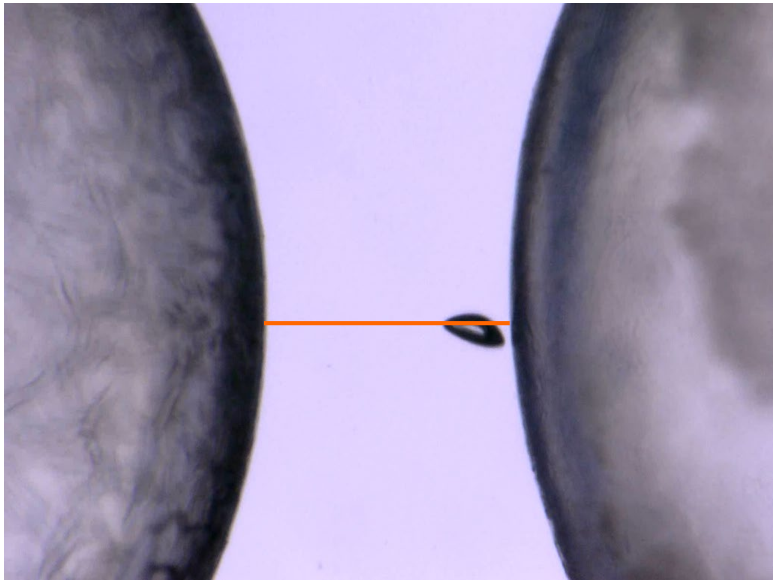
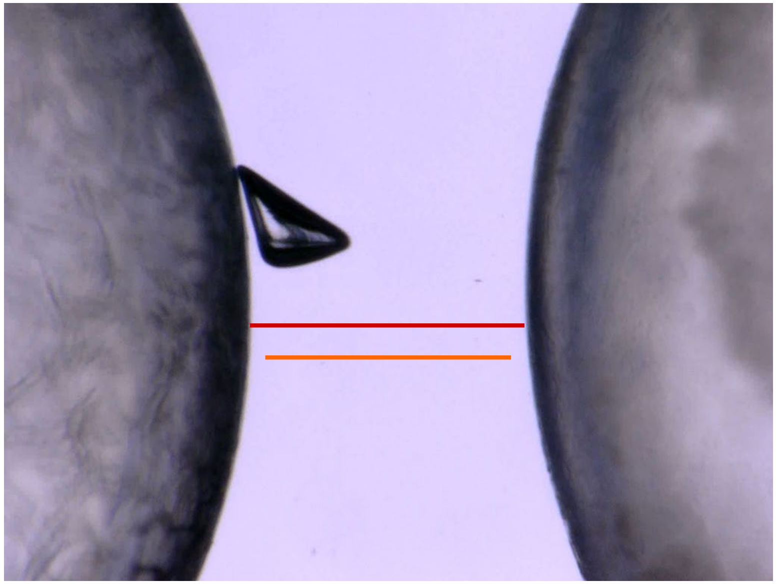
The Gent-Park experiment revisited

1st cycle



2nd cycle



In summary:

1. **Nucleation**: the nucleation of fracture — a.k.a. cavitation — occurs through the growth of submicron defects into cavities/cracks of micron size and roughly spherical deformed shape in regions where the hydrostatic-to-shear stress ratio is sufficiently high

2. **Propagation (micro)**: the nucleated cavities/cracks proceed their growth by fracture into cracks of tens of microns in size — which we term micro-cracks — and thus become visible optically. They do so, however, at a markedly lower rate than that observed during their nucleation

3. **Propagation (macro) & healing**: one among the various micro-cracks grows by fracture into a crack of hundreds of microns in size — which we term macro-cracks — while the other micro-cracks stop growing, then decrease in size, and eventually completely heal

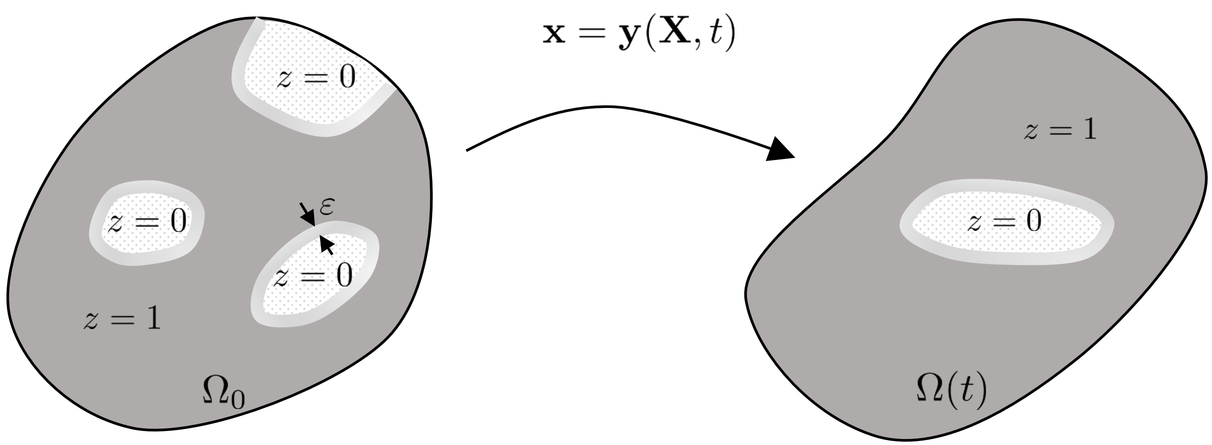
4. **Evolution of the fracture & healing properties**: regions of the elastomer that experience healing seem to acquire different fracture and healing properties from the original those of the original elastomer

A Phase-Transition Theory of Fracture & Healing of Elastomers

The two-pronged basic idea

1. Consider elastomers as solids **capable to undergo finite deformations** and **capable** also **to phase transition** to another solid of vanishingly small stiffness

2. Take the phase transition to be driven by the **competition** between a combination of strain energy AND stress concentration in the **bulk** and surface energy on the created/healed new **surfaces** in the elastomer



Configurational variable $z(\mathbf{X},t)$ $\begin{cases} z=1 \text{ } \textit{phase 1} \text{ (the solid)} \\ z=0 \text{ } \textit{phase 2} \text{ (the fractured elastomer)} \end{cases}$

The transition from one phase to the other is not sharp, but rather, diffuse over a thin interphase of size ε where $0<z<1$

Ziegler (1958)

Fried and Gurtin (1994); Gurtin (1996)

The framework of configurational forces (*a la* two potentials)

Thermodynamic potentials

Free-energy function: $\boxed{\psi = \psi(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha})}$

Dissipation potential: $\boxed{\phi = \phi(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha}, \dot{z}, \dot{\mathbf{Z}}, \dot{\boldsymbol{\alpha}})}$

Here, $\mathbf{F} = \operatorname{Grad} \mathbf{y}$

$\mathbf{Z} = \operatorname{Grad} z$

$\boldsymbol{\alpha}$ – internal variables

Constitutive relations

$$\mathbf{S}(\mathbf{X}, t) = \frac{\partial \psi}{\partial \mathbf{F}}(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha})$$ – 1st Piola-Kirchhoff stress

$$c_{\mathtt{i}}(\mathbf{X}, t) = -\frac{\partial \psi}{\partial z}(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha}) - \frac{\partial \phi}{\partial \dot{z}}(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha}, \dot{z}, \dot{\mathbf{Z}}, \dot{\boldsymbol{\alpha}})$$ – configurational *internal* force

$$\mathbf{C}(\mathbf{X}, t) = \frac{\partial \psi}{\partial \mathbf{Z}}(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha}) + \frac{\partial \phi}{\partial \dot{\mathbf{Z}}}(\mathbf{F}, z, \mathbf{Z}, \boldsymbol{\alpha}, \dot{z}, \dot{\mathbf{Z}}, \dot{\boldsymbol{\alpha}})$$ – configurational stress

$$\boldsymbol{\mathcal{A}}\{\dot{\boldsymbol{\alpha}}, \boldsymbol{\alpha}; \mathbf{F}, z, \mathbf{Z}\} = \mathbf{0}$$ – evolution equation for the internal variables $\boldsymbol{\alpha}$

Boundary data and source terms (external stimuli)

$$\boxed{\begin{aligned} &\mathbf{y} = \boldsymbol{\xi}(\mathbf{X}, t) \ \text{ on } \ \partial\Omega_0^{\mathbf{y}} \times [0, T] \\ &\mathbf{S}\mathbf{N} = \boldsymbol{\sigma}(\mathbf{X}, t) \ \text{ on } \ \partial\Omega_0^{\mathbf{S}} \times [0, T] \end{aligned}}$$

$$\boxed{\mathbf{C} \cdot \mathbf{N} = 0 \ \text{ on } \ \partial\Omega_0 \times [0, T]}$$

and

$\boxed{\mathbf{b}(\mathbf{X}, t) \ \text{ in } \ \Omega_0 \times [0, T]}$ – body force

$\boxed{c_{\mathtt{e}}(\mathbf{X}, t) \ \text{ in } \ \Omega_0 \times [0, T]}$ – configurational *external* force

The framework of configurational forces (*a la* two potentials)

Balance principles

$$\boxed{\mathrm{Div}\,\mathbf{S} + \mathbf{b} = \mathbf{0}}$$

Balance of Linear Momentum

$$\boxed{\mathrm{Div}\,\mathbf{C} + c_{\mathsf{i}} + c_{\mathsf{e}} = 0}$$

Balance of Configurational Forces

Governing equations (putting things together)

$$\begin{cases} \mathrm{Div}\left[\dfrac{\partial\psi}{\partial\mathbf{F}}\right] + \mathbf{b} = \mathbf{0} & \text{in } \Omega_0 \times [0,T] \\ \mathbf{y} = \boldsymbol{\xi} & \text{on } \Omega_0^{\mathbf{y}} \times [0,T] \\ \dfrac{\partial\psi}{\partial\mathbf{F}}\mathbf{N} = \boldsymbol{\sigma} & \text{on } \Omega_0^{\mathbf{S}} \times [0,T] \end{cases}$$

$$\begin{cases} \mathrm{Div}\left[\dfrac{\partial\psi}{\partial\mathbf{Z}} + \dfrac{\partial\phi}{\partial\dot{\mathbf{Z}}}\right] + c_{\mathsf{e}} = \dfrac{\partial\psi}{\partial z} + \dfrac{\partial\phi}{\partial\dot{z}} & \text{in } \Omega_0 \times [0,T] \\ \left[\dfrac{\partial\psi}{\partial\mathbf{Z}} + \dfrac{\partial\phi}{\partial\dot{\mathbf{Z}}}\right]\cdot\mathbf{N} = 0 & \text{on } \partial\Omega_0 \times [0,T] \\ z(\mathbf{X},0) = z_0(\mathbf{X}), & \mathbf{X} \in \Omega_0 \end{cases}$$

along with

$$\boxed{\boldsymbol{\mathcal{A}}\{\dot{\boldsymbol{\alpha}},\boldsymbol{\alpha};\mathbf{F},z,\mathbf{Z}\} = \mathbf{0} \quad \text{in } \Omega_0 \times [0,T] \qquad \text{with} \qquad \boldsymbol{\alpha}(\mathbf{X},0) = \boldsymbol{\alpha}_0(\mathbf{X}), \quad \mathbf{X} \in \Omega_0}$$

for the deformation field $\mathbf{y}(\mathbf{X},t)$

the configurational field $z(\mathbf{X},t)$

the internal variables $\boldsymbol{\alpha}(\mathbf{X},t)$

The Proposed Theory

The theory: the free energy

Free-energy function:

$$\psi(\mathbf{F}, z, \mathbf{Z}, \alpha) = \underbrace{(z^2 + \eta)\, W(\mathbf{F}) + (z^2 + \eta_\kappa)\, \kappa\, g(\mathbf{F})}_{\textit{elastic energy storage}} + \ldots$$

The original elastomer ($z = 1$):

$$W_{\mathcal{O}} = W(\mathbf{F}) + \kappa\, g(\mathbf{F}) \quad \text{with} \quad W_{\mathcal{O}} = \begin{cases} W(\mathbf{F}) & \text{if } \det \mathbf{F} = 1 \\ +\infty & \text{otherwise} \end{cases} \quad \text{for} \quad \kappa = +\infty$$

The fractured elastomer ($z = 0$):

$$W_{\mathcal{F}} = \eta W(\mathbf{F}) + \eta_\kappa \kappa\, g(\mathbf{F}) \quad \text{with} \quad \eta_\kappa \leq \eta \ll 1$$

The theory: the free energy

Free-energy function:

$$\psi(\mathbf{F}, z, \mathbf{Z}, \alpha) = \underbrace{(z^2 + \eta)\, W(\mathbf{F}) + (z^2 + \eta_\kappa)\, \kappa\, g(\mathbf{F})}_{\textit{elastic energy storage}} + \ldots$$

The theory: the free energy

Free-energy function:

$$\psi(\mathbf{F}, z, \mathbf{Z}, \alpha) = (z^2 + \eta)\, W(\mathbf{F}) + (z^2 + \eta_\kappa)\, \kappa\, g(\mathbf{F}) + \underbrace{k_{\mathcal{S}}(\alpha)\, \frac{3}{8} \left(\frac{1 - z}{\varepsilon} + \varepsilon\, \mathbf{Z} \cdot \mathbf{Z} \right)}_{\textit{surface energy storage}}$$

Here, $k_{\mathcal{S}}(\alpha)$ is a material function of the cumulative history of fracture and healing as measured by the **internal variable**

$$\alpha(\mathbf{X}, t) = \int_0^t |\dot{z}(\mathbf{X}, \tau)|\, \mathrm{d}\tau$$

Remark: Consistent with the pioneering postulate of Griffith (1921), in that the ***surface energy stored*** in the elastomer is taken to be proportional to the surface area that the fractured elastomer occupies

Braides (1998); Pham, Amor, Marigo, Maurini (2011)

The theory: the free energy & the dissipation potential

Free-energy function:

$$\psi(\mathbf{F}, z, \mathbf{Z}, \alpha) = (z^2 + \eta)\, W(\mathbf{F}) + (z^2 + \eta_\kappa)\, \kappa\, g(\mathbf{F}) + k_{\mathcal{S}}(\alpha)\, \frac{3}{8} \left(\frac{1-z}{\varepsilon} + \varepsilon\, \mathbf{Z} \cdot \mathbf{Z} \right)$$

Dissipation potential:

$$\phi(z, \mathbf{Z}, \alpha, \dot{z}, \dot{\mathbf{Z}}) = \begin{cases} \dfrac{3\, k_{\mathcal{F}}(\alpha)}{8} \left(-\dfrac{\dot{z}}{\varepsilon} + 2\varepsilon\, \mathbf{Z} \cdot \dot{\mathbf{Z}} \right) & \text{if} \quad \dot{z} \leq 0 \\[2ex] -\dfrac{3\, k_{\mathcal{H}}(\alpha)}{8} \left(-\dfrac{\dot{z}}{\varepsilon} + 2\varepsilon\, \mathbf{Z} \cdot \dot{\mathbf{Z}} \right) & \text{if} \quad \dot{z} > 0 \end{cases}$$

Remark: The dissipation potential is also consistent with the pioneering postulate of Griffith (1921), in that the ***surface energy dissipated*** in fracturing or healing is taken to be proportional to the surface area that is created or healed

The theory: the configurational external force

Configurational external force:

$$c_{\mathsf{e}}(\mathbf{X},t) = -\underbrace{\gamma}_{0 \text{ or } O(1)}\, z \,\frac{\overbrace{3^{\frac{p}{2}}}^{\textit{strain measure}}}{|\mathbf{F}|^p}\underbrace{\left(\frac{\kappa}{3\,\det \mathbf{F}}\,\mathbf{F}\cdot\frac{\partial g}{\partial \mathbf{F}}(\mathbf{F})\right)}_{\textit{hydrostatic part of the Cauchy stress}}$$

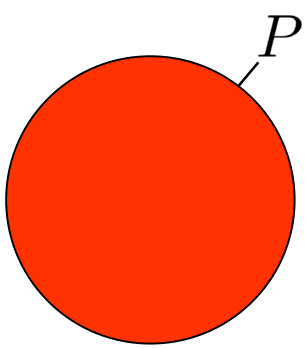
The theory: the configurational external force

Configurational external force:

$$c_{\mathsf{e}}(\mathbf{X},t) = -\gamma\, z\, \frac{3^{\frac{p}{2}}}{|\mathbf{F}|^p} \left(\frac{\kappa}{3\,\det \mathbf{F}}\, \mathbf{F} \cdot \frac{\partial g}{\partial \mathbf{F}}(\mathbf{F}) \right)$$

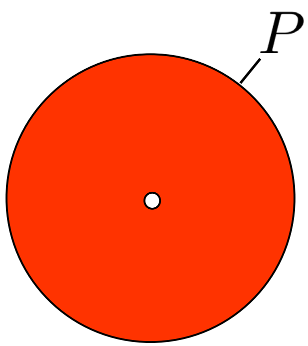
Physical interpretation: **Griffith's idea is fundamentally incomplete!**

***macro*scopic view**



ball of an incompressible elastomer
under hydrostatic loading

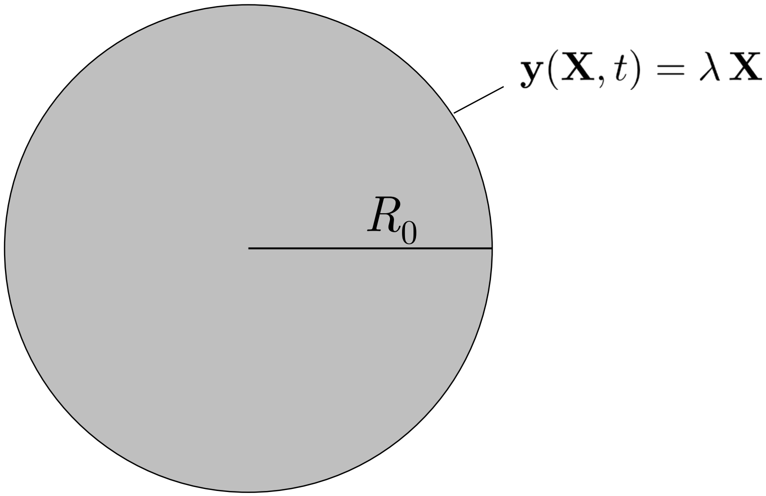
***micro*scopic view**



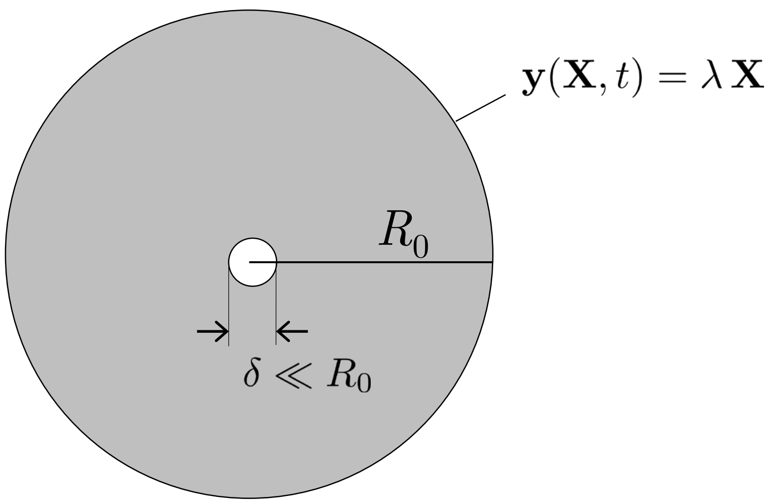
ball of an incompressible elastomer,
containing a defect,
under hydrostatic loading

nucleation of a macroscopic crack = propagation of a microscopic defect

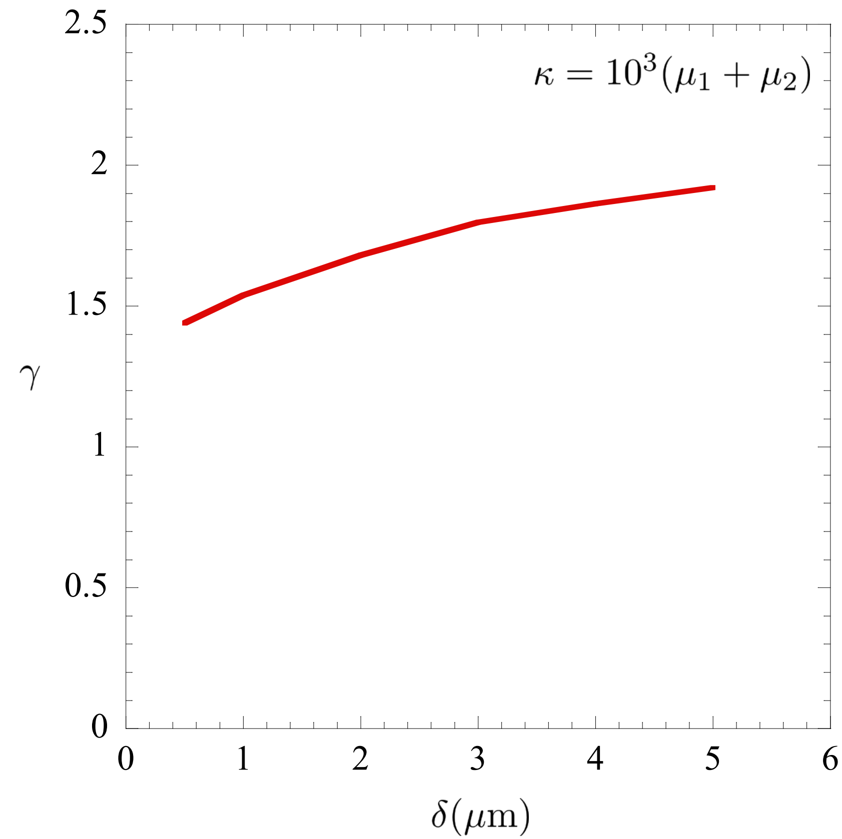
The stochastic nature of γ



***macro*scopic view**



***micro*scopic view**



The governing equations

Putting things together, we get the coupled boundary-value problems

$$\begin{cases} \mathrm{Div}\left[(z^2+\eta)\dfrac{\partial W}{\partial \mathbf{F}} + (z^2+\eta_\kappa)\kappa\dfrac{\partial g}{\partial \mathbf{F}}\right] = \mathbf{0}, & (\mathbf{X},t)\in\Omega_0\times[0,T] \\ \mathbf{y}(\mathbf{X},t) = \boldsymbol{\xi}(\mathbf{X},t), & (\mathbf{X},t)\in\partial\Omega_0^{\mathbf{y}}\times[0,T] \\ \left[(z^2+\eta)\dfrac{\partial W}{\partial \mathbf{F}} + (z^2+\eta_\kappa)\kappa\dfrac{\partial g}{\partial \mathbf{F}}\right]\mathbf{N} = \sigma(\mathbf{X},t), & (\mathbf{X},t)\in\partial\Omega_0^{\mathbf{S}}\times[0,T] \end{cases}$$

and

$$\begin{cases} \mathrm{Div}\left[\varepsilon k(\alpha)\mathbf{Z}\right] = \dfrac{8}{3}z\left(W+\kappa g\right) + 4\gamma\, z\,\dfrac{3^{\frac{p-4}{2}}}{|\mathbf{F}|^p}\left(\dfrac{\kappa}{\det\mathbf{F}}\,\mathbf{F}\cdot\dfrac{\partial g}{\partial \mathbf{F}}\right) - \dfrac{k(\alpha)}{2\varepsilon}, & (\mathbf{X},t)\in\Omega_0\times[0,T] \\ \mathbf{Z}\cdot\mathbf{N} = 0, & (\mathbf{X},t)\in\partial\Omega_0\times[0,T] \\ z(\mathbf{X},0) = z_0(\mathbf{X}), & \mathbf{X}\in\Omega_0 \end{cases}$$

for the deformation field $\mathbf{y}(\mathbf{X},t)$ and the configurational field $z(\mathbf{X},t)$

Remark 1: the toughness function $k(\alpha)$

The toughness function that has emerged in the governing PDEs is given by

$$k(\alpha) = \begin{cases} k_{\mathcal{S}}(\alpha) + k_{\mathcal{F}}(\alpha) = k^0_{\mathcal{F}} + H_{\mathcal{F}}\left(\alpha(\mathbf{X}, t^*(\mathbf{X}, t))\right) & \text{if} \quad \dot{z} \leq 0 \\ k_{\mathcal{S}}(\alpha) - k_{\mathcal{H}}(\alpha) = k^0_{\mathcal{H}} + H_{\mathcal{H}}\left(\alpha(\mathbf{X}, t^*(\mathbf{X}, t))\right) & \text{if} \quad \dot{z} > 0 \end{cases}$$

where

$$\alpha(\mathbf{X}, t) = \int_0^t |\dot{z}(\mathbf{X}, \tau)| \, \mathrm{d}\tau \qquad \text{and} \qquad t^*(\mathbf{X}, t) = \sup\left\{\tau : \tau < t \ \text{ and } \ z(\mathbf{X}, \tau) = 1\right\}$$

physically, it is a measure of the material resistance to fracture and to heal

Note: $k^0_{\mathcal{F}} = G_c \geq 0$ for standard elastomers $0.05\,\mathrm{J}/\mathrm{m}^2 \leq G_c \leq 1000\,\mathrm{J}/\mathrm{m}^2$

Note: $k^0_{\mathcal{H}} \leq k^0_{\mathcal{F}}$ with equality holding only when the processes of fracturing and healing do *not* incur energy dissipation

Note: if $k(\alpha) \leq 0$ for the healing branch, then healing is prohibited (i.e., fracture is irreversible)

Lakes and Thomas (1967); Gent and Wang (1991); Fond et al. (1996)

Remark 2: the competing *bulk* and *surface* quantities

The right-hand side of the PDE for the configurational variable

$$\begin{cases} \mathrm{Div}\,[\varepsilon k(\alpha)\mathbf{Z}] = \boxed{\dfrac{8}{3} z\,(W + \kappa g) + 4\gamma\, z\, \dfrac{3^{\frac{p-4}{2}}}{|\mathbf{F}|^p} \left(\dfrac{\kappa}{\det \mathbf{F}}\, \mathbf{F} \cdot \dfrac{\partial g}{\partial \mathbf{F}} \right) - \dfrac{k(\alpha)}{2\varepsilon},} & (\mathbf{X}, t) \in \Omega_0 \times [0, T] \\ \mathbf{Z} \cdot \mathbf{N} = 0, & (\mathbf{X}, t) \in \partial\Omega_0 \times [0, T] \\ z(\mathbf{X}, 0) = z_0(\mathbf{X}), & \mathbf{X} \in \Omega_0 \end{cases}$$

makes apparent the precise competition between bulk and surface quantities that drive the nucleation and propagation of fracture and healing in the elastomer:

Nucleation and propagation of fracture will occur at a material point X whenever the strain energy and ratio of hydrostatic stress to stretch is sufficiently large relative to the fracture toughness of the material

By the same token, healing will occur if the same combination of strain energy and stress is sufficiently small relative to the healing toughness of the material

Remark 3: the Francfort & Marigo connection

By construction, for the case when

$$\gamma = 0, \qquad k_{\mathcal{S}}(\alpha) = k_{\mathcal{H}}(\alpha) = 0, \qquad k_{\mathcal{F}}(\alpha) = k^0_{\mathcal{F}} = G_c, \qquad \eta_\kappa = \eta$$

the proposed governing equations reduce essentially to Euler-Lagrange equations of the celebrated regularized variational theory of brittle fracture by Francfort & Marigo (1998), as put forth by Bourdin, Francfort, & Marigo (2008).

The proposed theory can thus be thought of as a natural generalization of the Francfort and Marigo theory to account for the three more general physical attributes innate to elastomers:

i) the nucleation of fracture does not necessarily occur in regions where the strain energy concentrates, instead, because of the typical near incompressibility of elastomers, fracture may nucleate in regions where the (hydrostatic) stress concentrates

ii) fracture is not necessarily a purely dissipative and irreversible process, instead, fractured surfaces may store energy and healing (partial or complete) is allowed, and

iii) the resistance of elastomers to fracture and to heal is not characterized by a material constant, instead, it is characterized by a material function dependent on the cumulative history of fracture and healing.

Remark 4: the Griffith connection

VI. *The Phenomena of Rupture and Flow in Solids.*

By A. A. GRIFFITH, *M. Eng.* (*of the Royal Aircraft Establishment*).

Communicated by G. I. TAYLOR, *F.R.S.*

Received February 11,—Read February 26, 1920.

The calculation of the potential energy is facilitated by the use of a general theorem which may be stated thus: In an elastic solid body deformed by specified forces applied at its surface, the sum of the potential energy of the applied forces and the strain energy of the body is diminished or unaltered by the introduction of a crack whose surfaces are traction-free.

This theorem may be proved* as follows: It may be supposed, for the present purpose, that the crack is formed by the sudden annihilation of the tractions acting on its surface. At the instant following this operation, the strains, and therefore the potential energy under consideration, have their original values; but, in general, the new state is not one of equilibrium. If it is not a state of equilibrium, then, by the theorem of minimum energy, the potential energy is reduced by the attainment of equilibrium; if it is a state of equilibrium the energy does not change. Hence the theorem is proved.

A Robust and Efficient Numerical Implementation of the Theory

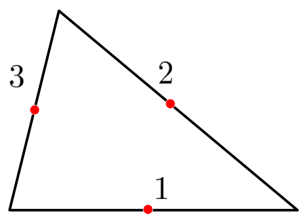
Numerical implementation

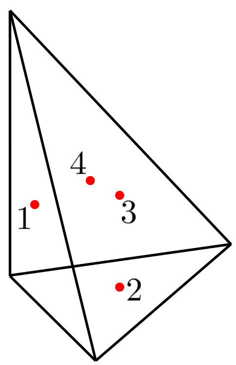
Key ingredients:

- First-order time discretization

$$[0,T] \to 0 = t^0, t^1, \ldots t^m, t^{m+1}, \ldots, t^M = T$$

- Space discretization based on non-conforming finite elements of low order





FE discretization capable of dealing with:

- **finite deformations**
- **nearly incompressible elastomers**
- **using fewer degrees of freedom than competing hybrid discretizations**

- Gradient flow solver based on a staggered strategy

Newton's method, or any of its variations, does not seem capable of delivering converged solutions for general specimen geometries and loading conditions

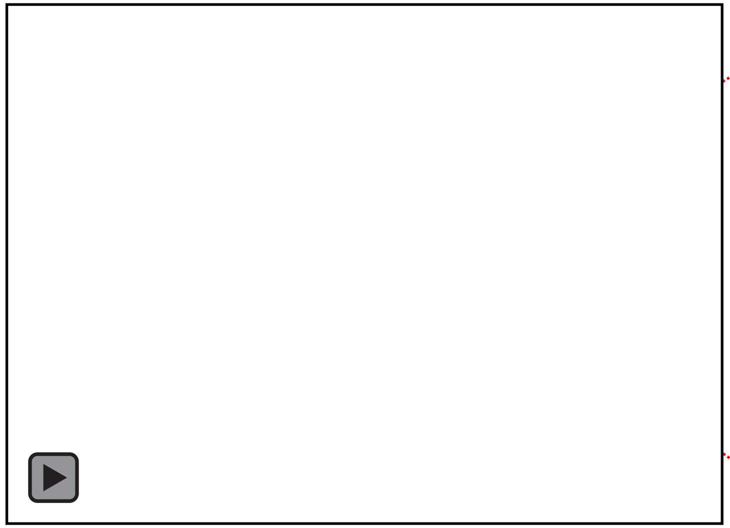
Cauchy (1847); Crouzeix & Raviart (1973); Behrman (1998)

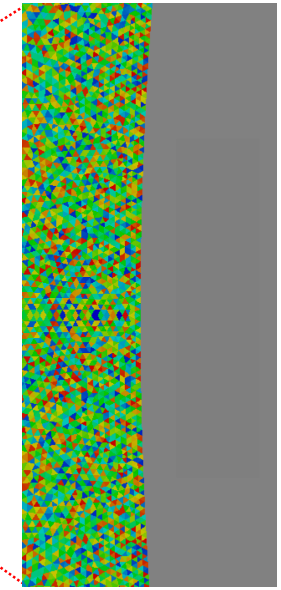
Comparisons With the Gent-Park Experiments of Poulain et al. (2017; 2018)

Comparisons with experiments

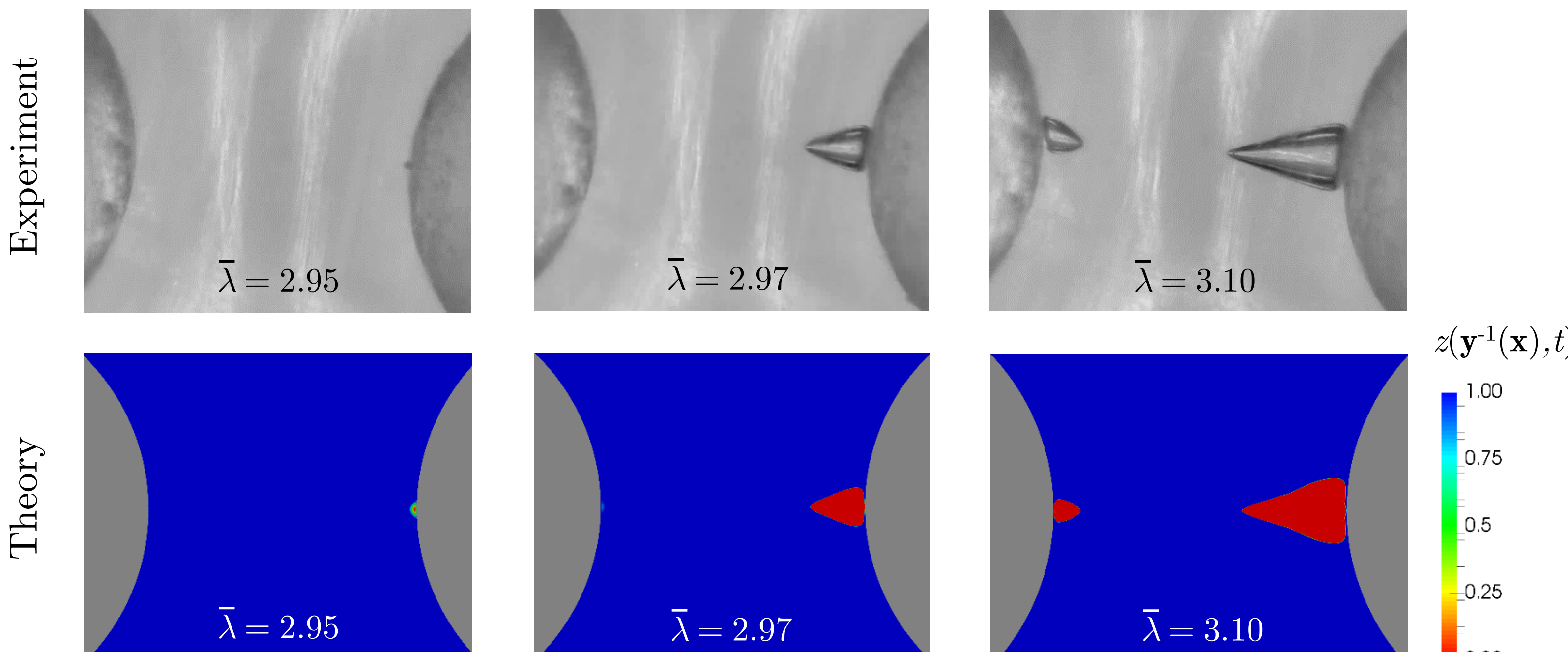
PDMS 45:1,

$$\frac{H}{D} = 0.210$$





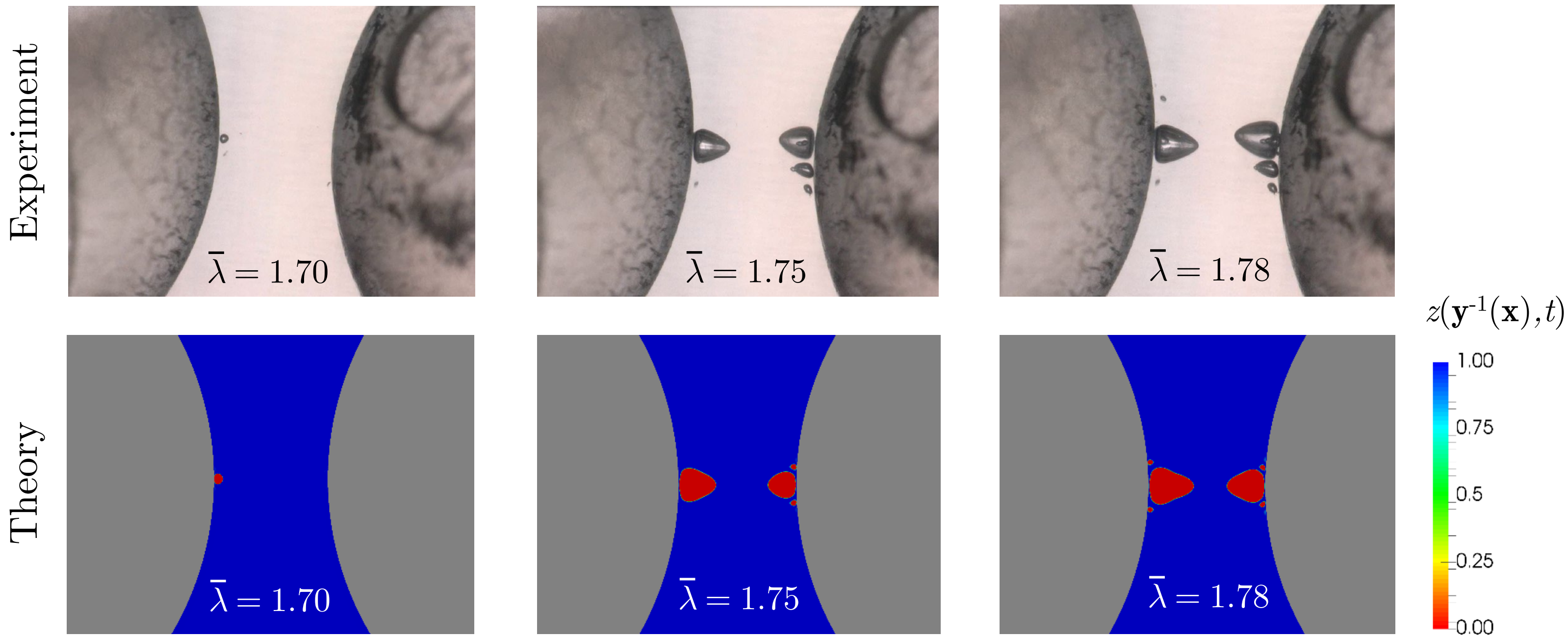
γ



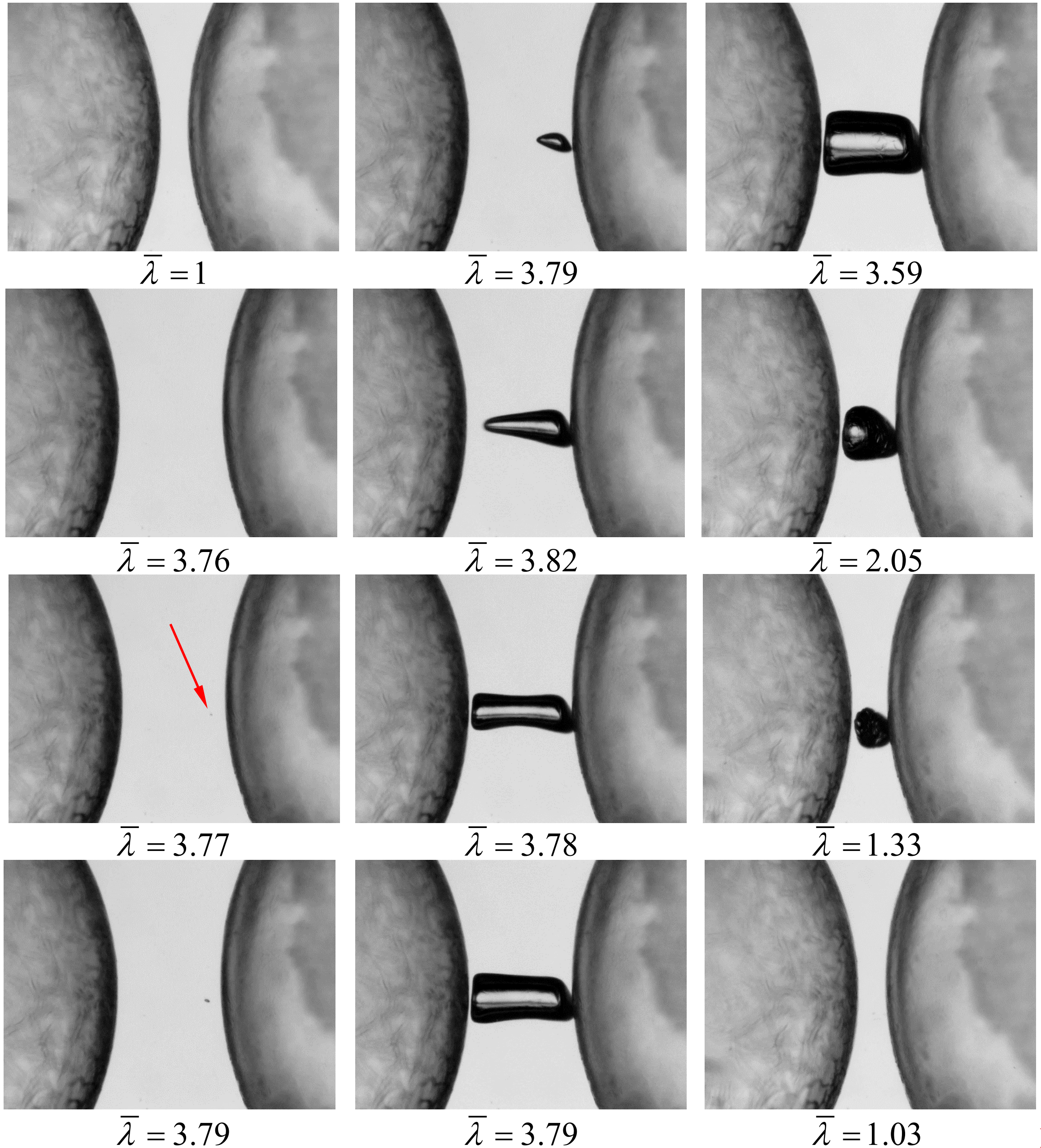
Comparisons with experiments

PDMS 15:1,

$$\frac{H}{D} = 0.107$$



Comparisons with experiments

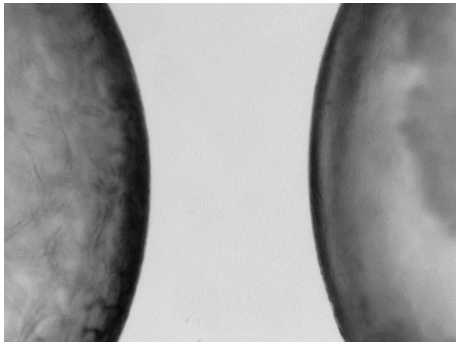


1st cycle

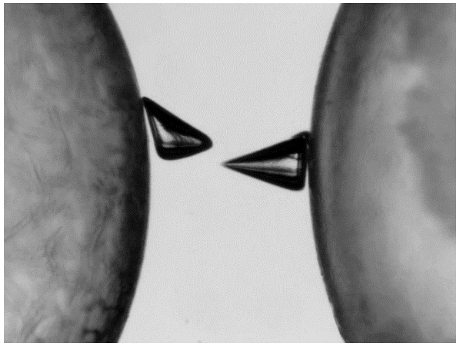
PDMS 30:1, $\frac{H}{D}=0.051$

Kumar, Ravi-Chandar, Lopez-Pamies *Int. J. Frac.* (2018)

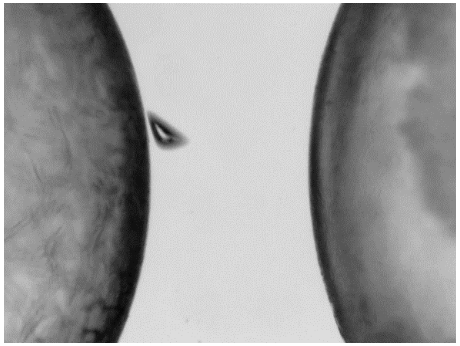
Comparisons with experiments



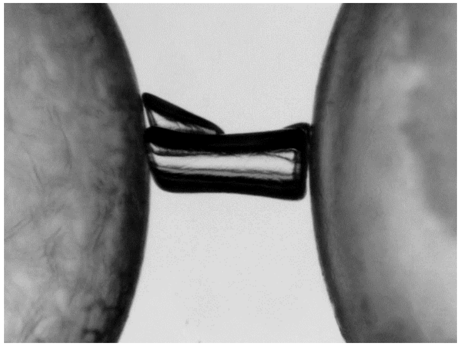
$\bar{\lambda} = 4.24$



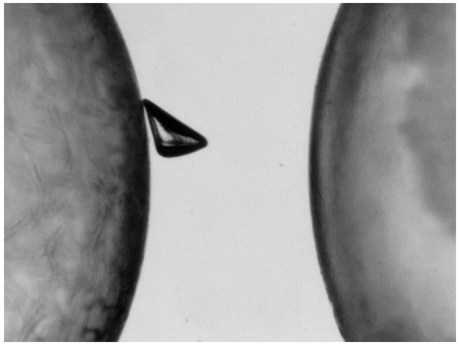
$\bar{\lambda} = 4.27$



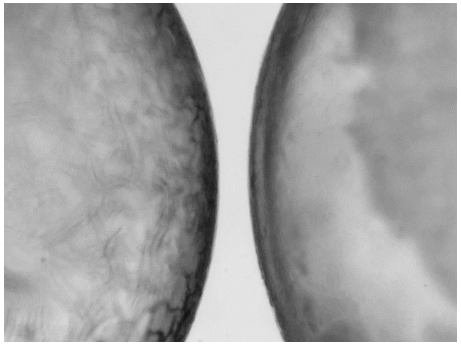
$\bar{\lambda} = 4.26$



$\bar{\lambda} = 4.27$



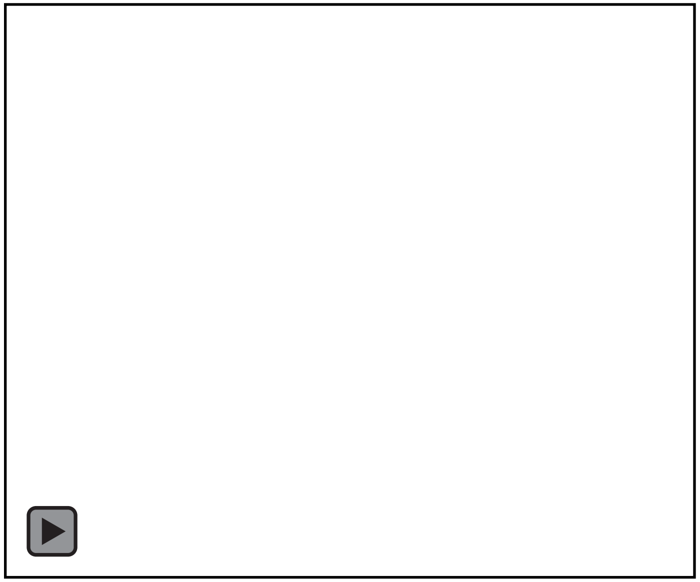
$\bar{\lambda} = 4.27$



$\bar{\lambda} = 0.97$

2nd cycle

PDMS 30:1, $\frac{H}{D} = 0.051$



Kumar, Ravi-Chandar, Lopez-Pamies *Int. J. Frac.* (2018)