The elastic dielectric response of elastomers filled with liquid inclusions: From fundamentals to governing equations

Oscar Lopez-Pamies

Department of Civil and Environmental Engineering
University of Illinois, Urbana–Champaign, IL 61801, USA

Abstract. Over the past decade, soft solids containing electro- and magneto-active *liquid* — as opposed to solid — inclusions have emerged as a new class of smart materials with promising novel electromechanical properties. In this context, a recent contribution has put forth a continuum theory that describes the macroscopic elastic behavior of elastomers filled with liquid inclusions under quasistatic finite deformations from the bottom up, directly in terms of their microscopic behavior at the length scale of the inclusions. This chapter presents the generalization of that theory to the coupled realm of the elastic dielectric behavior of such an emerging class of filled elastomers when in addition to undergoing quasistatic finite deformations they are subjected to quasistatic electric fields. The chapter starts with the description of the underlying fundamentals in the continuum — *id est*, kinematics, conservation of mass, Maxwell's equations, balance of momenta, and constitutive behavior of both the bulk (the solid elastomer and the liquid inclusions) and the solid/liquid interfaces — and ends with their combination to formulate the resulting governing equations.

Contents

1 Introduction

The study of the mechanics of interfaces in the continuum has a long and rich history with origins dating back to the classical works of Young (1805) and Laplace (1806) in the early 1800s on interfaces between fluids and of Gibbs (1878) in the 1870s on the more general case of interfaces between solids and fluids. Despite this early origins, it was only in 1975 that complete descriptions of the kinematics, the concept of interface stress, and the balance of linear and angular momenta of bodies containing material interfaces were properly formulated, alongside their specialization to the basic constitutive case of elastic interfaces (Gurtin and Murdoch (1975a,b)). This advancement in theory, however, was not followed by its exploitation in practice, surely because of the technical difficulties of measuring and tailoring the mechanical properties of interfaces at the time. This changed in the early 2000s, when the appearance of new synthesis and characterization tools reinvigorated the study of interfaces in soft matter.

In this context, elastomers filled with *liquid* — as opposed to solid — inclusions are a recent trend in the soft matter community because — thanks

to the behavior of the solid/liquid interfaces — they are capable of exhibiting remarkable mechanical and physical properties; see, e.g., Lopez-Pamies (2014); Style et al. (2015a); Bartlett et al. (2017); Lefèvre and Lopez-Pamies (2017a,b); Lefèvre et al. (2017); Yun et al. (2019). Indeed, the interfacial mechanics in these soft material systems can be actively tailored to enhance or impede deformability. In particular, while the addition of liquid inclusions should increase the macroscopic deformability of the material, the behavior of the solid/liquid interfaces, if negligible when the inclusions are "large", may counteract the macroscopic properties of the material when the inclusions become sufficiently "small".

As a first step to understand in a precise and quantitative manner this new paradigm, Ghosh and Lopez-Pamies (2022) have recently worked out the governing equations that describe from the bottom up the mechanical response of an elastic solid filled with initially spherical inclusions made of a pressurized elastic fluid when the solid/fluid interfaces are elastic and possess an initial surface tension; see also Ghosh et al. (2023a,b) for applications, Casado Díaz et al. (2023) for the associated mathematical analysis, as well as Style et al. (2015b), Wang and Henann (2016), and Krichen et al. (2019) for earlier preliminary studies.

In this chapter, we generalize the formulation of Ghosh and Lopez-Pamies (2022) to the coupled realm of elastic dielectric behavior. In particular, we work out the governing equations that describe the electromechanical response of an elastic dielectric solid filled with initially spherical inclusions made of a pressurized elastic dielectric fluid when the solid/fluid interfaces feature their own elastic dielectric behavior and possess an initial surface tension.

The organization of the chapter is as follows. In Sections 2 through 6, we present separately the relevant basic ingredients of:

- initial configuration and kinematics of the bulk and interfaces,
- conservation of mass,
- Maxwell's equations in the presence of material interfaces,
- balance of linear and angular momenta in the presence of material interfaces, and
- elastic dielectric constitutive behavior of the bulk and interfaces.

The combination of these ingredients leads to the governing equations that describe the elastic dielectric response of elastomers filled with liquid inclusions under finite quasistatic deformations and quasistatic electric fields. We present these in Section 7.

2 Initial configuration and kinematics

Initial configuration Consider a body made of M liquid inclusions fully embedded in a solid matrix that in its initial configuration occupies the open domain $\Omega_0 \subset \mathbb{R}^3$, with boundary $\partial\Omega_0$ and outward unit normal $\mathbf{N}$. Denote by $\Omega_0^{\mathtt{m}}$ the subdomain occupied by the $\mathtt{matrix}$ and by $\Omega_0^{\mathtt{i},j}$ $j = 1, 2, ..., M$ that occupied by the jth $\mathtt{inclusion}$. The inclusions are separated from the matrix by smooth interfaces, denoted by Γ_0^j for the jth inclusion, with unit normal $\widehat{\mathbf{N}}$ pointing outwards from the inclusions towards the matrix, so that $\Omega_0 = \Omega_0^{\mathtt{m}} \bigcup \Gamma_0 \bigcup \Omega_0^{\mathtt{i}}$, where $\Gamma_0 = \bigcup_{j=1}^{M} \Gamma_0^j$ and $\Omega_0^{\mathtt{i}} = \bigcup_{j=1}^{M} \Omega_0^{\mathtt{i},j}$. We identify material points in the body by their initial position vector

$$\mathbf{X} \in \Omega_0$$

and denote by $\theta_0^{\mathtt{i},j}(\mathbf{X})$ and $\theta_0^{\mathtt{i}}(\mathbf{X})$ the characteristic or indicator functions describing the individual and collective spatial locations occupied by the inclusions in Ω_0, that is,

$$\theta_0^{\mathtt{i},j}(\mathbf{X}) = \begin{cases} 1 & \text{if } \mathbf{X} \in \Omega_0^{\mathtt{i},j} \\ 0 & \text{otherwise} \end{cases} \qquad j = 1, 2, ..., M \quad \text{and} \quad \theta_0^{\mathtt{i}}(\mathbf{X}) = \sum_{j=1}^{M} \theta_0^{\mathtt{i},j}(\mathbf{X}). \tag{1}$$

As will become apparent below, it is convenient to single out the material points on the interfaces with their own labeling. We write

$$\widehat{\mathbf{X}} = \mathbf{X} \quad \text{when} \quad \mathbf{X} \in \Gamma_0.$$

Figure 1(a) shows a schematic of the body in its initial configuration with all the pertinent geometric quantities depicted.

Kinematics In response to the externally applied mechanical forces and electric fields described further below, the position vector $\mathbf{X}$ of a material point may occupy a new position $\mathbf{x}$ specified by a continuous[1], invertible, orientation-preserving mapping $\mathbf{y}$ from Ω_0 to the current configuration $\Omega = \Omega^{\mathtt{m}} \bigcup \Gamma \bigcup \Omega^{\mathtt{i}} \subset \mathbb{R}^3$, termed the *deformation field*. Here, in direct analogy to their initial counterparts, $\Omega^{\mathtt{m}}$, $\Omega^{\mathtt{i}}$, and Γ denote the subdomains occupied by the matrix and the inclusions and the interfaces separating them; by the same token, the notation $\widehat{\mathbf{n}}$, $\theta^{\mathtt{i},j}$, and $\theta^{\mathtt{i}}$ is used to denote the counterparts of $\widehat{\mathbf{N}}$, $\theta_0^{\mathtt{i},j}$, and $\theta_0^{\mathtt{i}}$ in the current configuration. We write

$$\mathbf{x} = \mathbf{y}(\mathbf{X}).$$

[1] The focus here is on liquid inclusions, which naturally exhibit coherent interfaces with the surrounding solid matrix, thus our restriction to continuous deformation fields.

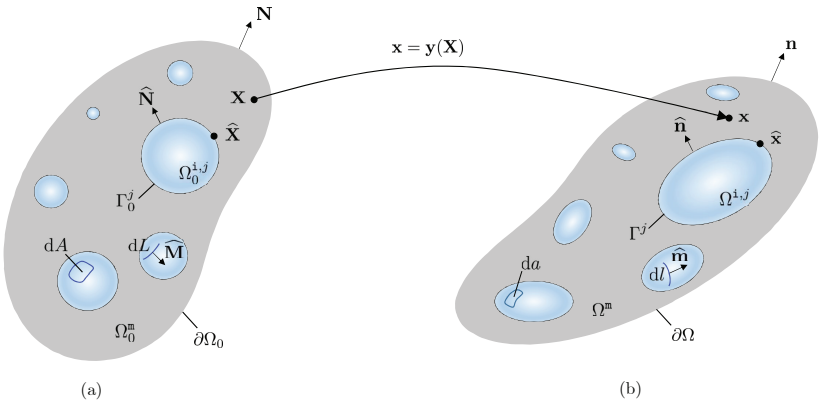

Figure 1. Schematics of (a) the initial and (b) the current configurations of a body made of a solid matrix filled with liquid inclusions.

Singling out again the material points on the interfaces with their own labeling, we also write

$$\widehat{\mathbf{x}} = \mathbf{y}(\widehat{\mathbf{X}}).$$

We denote the deformation gradient at $\mathbf{X} \in \Omega_0$ by

$$\mathbf{F}(\mathbf{X}) = \nabla \mathbf{y}(\mathbf{X}) = \frac{\partial \mathbf{y}}{\partial \mathbf{X}}(\mathbf{X})$$

and the *interface deformation gradient* at $\widehat{\mathbf{X}} \in \Gamma_0$ by

$$\widehat{\mathbf{F}}(\widehat{\mathbf{X}}) = \widehat{\nabla} \mathbf{y}(\widehat{\mathbf{X}}) = \mathbf{F}(\widehat{\mathbf{X}})\widehat{\mathbf{I}}, \tag{2}$$

where $\widehat{\mathbf{I}}$ stands for the projection tensor

$$\widehat{\mathbf{I}} = \mathbf{I} - \widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}.$$

The notation (2) merits some clarification. Assuming sufficient regularity away from the interfaces, the requirement that the deformation field $\mathbf{y}(\mathbf{X})$ be continuous implies the Hadamard jump condition

$$[\![\mathbf{F}(\widehat{\mathbf{X}})]\!]\widehat{\mathbf{I}} = \mathbf{0} \quad \text{with} \qquad [\![\mathbf{F}(\widehat{\mathbf{X}})]\!] := \mathbf{F}^{\mathtt{i}}(\widehat{\mathbf{X}}) - \mathbf{F}^{\mathtt{m}}(\widehat{\mathbf{X}}), \tag{3}$$

where $\mathbf{F}^{\mathtt{i}}$ ($\mathbf{F}^{\mathtt{m}}$) denotes the limit of $\mathbf{F}$ when approaching Γ_0 from $\Omega_0^{\mathtt{i}}$ ($\Omega_0^{\mathtt{m}}$). Although $\mathbf{F}^{\mathtt{i}} \neq \mathbf{F}^{\mathtt{m}}$ at Γ_0, $\mathbf{F}^{\mathtt{i}}\widehat{\mathbf{I}} = \mathbf{F}^{\mathtt{m}}\widehat{\mathbf{I}}$, and it is for this reason that, with

some abuse of notation, we do not include the label '$\mathtt{i}$' or '$\mathtt{m}$' in the right-hand side of (2). In the sequel, at solid/liquid interfaces, we always make use of the convention $[\![\cdot]\!] := (\cdot)^{\mathtt{i}} - (\cdot)^{\mathtt{m}}$ for the jump operator acting on any field.

The interested reader is referred to, for instance, do Carmo (2016); Weatherburn (2016); Gurtin et al. (1998); Javili et al. (2013) for a thorough description of differential operators defined on surfaces embedded in $\mathbb{R}^3$ and of the kinematics of interfaces. For our purposes here, it suffices to make explicit mention of some of the properties of the interface deformation gradient (2).

In direct analogy to the transformation rules for material line elements $\mathrm{d}\mathbf{X}$ in the bulk, material line elements $\mathrm{d}\widehat{\mathbf{X}}$ on the interfaces transform according to the rules

$$\mathrm{d}\widehat{\mathbf{x}} = \widehat{\mathbf{F}}\mathrm{d}\widehat{\mathbf{X}} \qquad \text{and} \qquad \mathrm{d}\widehat{\mathbf{X}} = \widehat{\mathbf{F}}^{-1}\mathrm{d}\widehat{\mathbf{x}}.$$

Owing to its rank deficiency, the inverse $\widehat{\mathbf{F}}^{-1}$ of the interface gradient deformation $\widehat{\mathbf{F}}$ is defined implicitly by the relations

$$\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}} = \widehat{\mathbf{I}} \qquad \text{and} \qquad \widehat{\mathbf{F}}\widehat{\mathbf{F}}^{-1} = \widehat{\mathbf{i}}.$$

where

$$\widehat{\mathbf{i}} = \mathbf{I} - \widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} \qquad \text{with} \qquad \widehat{\mathbf{n}} = \frac{1}{|J\mathbf{F}^{-T}\widehat{\mathbf{N}}|} J\mathbf{F}^{-T}\widehat{\mathbf{N}}.$$

That is

$$\widehat{\mathbf{F}}^{-1} = \mathbf{F}^{-1}\widehat{\mathbf{i}}.$$

In these last expressions, we have made use of the standard notation $J = \det \mathbf{F}$ for the determinant of the deformation gradient $\mathbf{F}$ and exploited the facts that $J^{\mathtt{i}}\mathbf{F}^{\mathtt{i}-T}\widehat{\mathbf{N}} = J^{\mathtt{m}}\mathbf{F}^{\mathtt{m}-T}\widehat{\mathbf{N}}$ and $\mathbf{F}^{\mathtt{i}-1}\widehat{\mathbf{i}} = \mathbf{F}^{\mathtt{m}-1}\widehat{\mathbf{i}}$, thanks to (3), to simply write, with the same abuse of notation as in (2), $J\mathbf{F}^{-T}\widehat{\mathbf{N}}$ and $\mathbf{F}^{-1}\widehat{\mathbf{i}}$ without the label '$\mathtt{i}$' or '$\mathtt{m}$'.

Furthermore, the area $\mathrm{d}A$ of material surface elements $\widehat{\mathbf{N}}\mathrm{d}A$ on the interfaces transforms according to the rule

$$\mathrm{d}a = \widehat{J}\mathrm{d}A \qquad \text{with} \qquad \widehat{J} = |J\mathbf{F}^{-T}\widehat{\mathbf{N}}|.$$

This transformation rule also serves to define the interface determinant operator $\widehat{\det}\,\widehat{\mathbf{F}} = \widehat{J}$.

Finally, we note that material curve elements $\widehat{\mathbf{M}}\,\mathrm{d}L$ on the interfaces transform according to the rule

$$\widehat{\mathbf{m}}\,\mathrm{d}l = \widehat{J}\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{M}}\,\mathrm{d}L,$$

where $\widehat{\mathbf{M}}$ is a unit vector that is tangential to Γ_0 and normal to the curve $\mathrm{d}L$. Figure 1(b) provides a schematic of the body in its current configuration with all the above geometric quantities depicted.

3 Conservation of mass

In any given subdomain of the current configuration $\mathcal{D} \subset \Omega$ we consider the existence of a *mass density*

$$\rho(\mathbf{x}) \geq 0, \quad \mathbf{x} \in \mathcal{D}.$$

Integral form Conservation of mass then reads

$$\int_{\mathcal{D}} \rho \,\mathrm{d}\mathbf{x} = \int_{\mathcal{D}_0} \rho_0 \,\mathrm{d}\mathbf{X}, \tag{4}$$

where $\rho_0(\mathbf{X})$ is the mass density in the initial configuration.

Localized form It follows from the integral form (4) that

$$\int_{\mathcal{D}_0} J\rho \,\mathrm{d}\mathbf{X} = \int_{\mathcal{D}_0} \rho_0 \,\mathrm{d}\mathbf{X}$$

and hence that conservation of mass can be rewritten in the localized form

$$\rho = J^{-1}\rho_0, \quad \mathbf{x} \in \Omega \setminus \Gamma. \tag{5}$$

4 Maxwell's equations in the presence of material interfaces

4.1 Bulk and interface charges, electric fields, and electric displacements

In any given subdomain of the current configuration $\mathcal{D} \subset \Omega$, with boundary $\partial\mathcal{D}$ and outward unit normal $\widetilde{\mathbf{n}}$, we consider the presence of a *space charge* density per unit current volume

$$q(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D},$$

an *electric displacement*

$$\mathbf{d}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D},$$

an *electric field*

$$\mathbf{e}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D},$$

an *interface charge* density per unit current area

$$\widehat{q}(\widehat{\mathbf{x}}), \quad \widehat{\mathbf{x}} \in \mathcal{S},$$

and an *interface electric displacement*

$$\widehat{\mathbf{d}}(\widehat{\mathbf{x}}), \quad \widehat{\mathbf{x}} \in \mathcal{S},$$

where $\mathcal{S} \subset \Gamma$, with boundary $\partial\mathcal{S}$, stands for any subsurfaces of the interfaces that the subdomain $\mathcal{D}$ may contain. The first four of these quantities are standard. The fifth one accounts for the possibility of an additional polarization mechanism at the matrix/inclusions interfaces. Within such a class of interface electric displacements, we shall restrict attention to *tangential* electric displacements in the sense that

$$\widehat{\mathbf{i}}\,\widehat{\mathbf{d}} = \widehat{\mathbf{d}}. \tag{6}$$

Figure 2 shows a schematic of a generic subdomain $\mathcal{D}$ with all five types of bulk and interface quantities depicted.

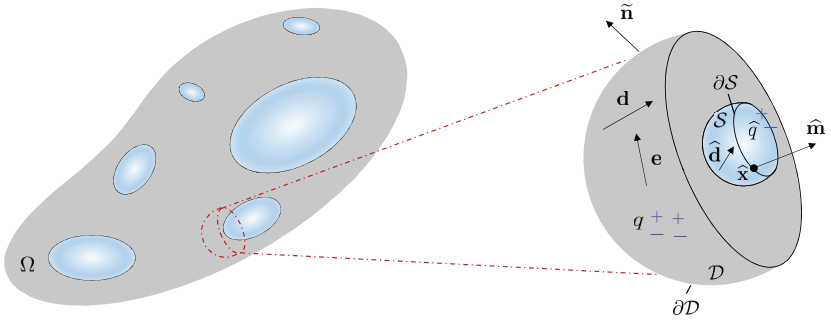

Figure 2. Schematic of a subdomain of the current configuration $\mathcal{D} \subset \Omega$, with boundary $\partial\mathcal{D}$ and outward unit normal $\widetilde{\mathbf{n}}$, indicating the space charge $q(\mathbf{x})$, the electric displacement $\mathbf{d}(\mathbf{x})$, the electric field $\mathbf{e}(\mathbf{x})$, the interface charge $\widehat{q}(\widehat{\mathbf{x}})$, and the interface electric displacement $\widehat{\mathbf{d}}(\widehat{\mathbf{x}})$.

4.2 Gauss's law

Integral form In view of the presence of bulk and interface charges and the interface electric displacement, the integral form of Gauss's law reads

$$\int_{\partial\mathcal{D}} \mathbf{d}\cdot\widetilde{\mathbf{n}}\,\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{S}} \widehat{\mathbf{d}}\cdot\widehat{\mathbf{m}}\,\mathrm{d}\widehat{\mathbf{x}} = \int_{\mathcal{D}} q\,\mathrm{d}\mathbf{x} + \int_{\mathcal{S}} \widehat{q}\,\mathrm{d}\widehat{\mathbf{x}}. \tag{7}$$

Localized form Making use of the bulk divergence theorem (written here, for clarity, in indicial notation with respect to a Cartesian frame of reference)

$$\int_{\mathcal{D}} \frac{\partial(\cdot)}{\partial x_k}\,\mathrm{d}\mathbf{x} = \int_{\partial\mathcal{D}} (\cdot)\widetilde{n}_k\,\mathrm{d}\mathbf{x} + \int_{\mathcal{S}} [\![\cdot]\!]\widehat{n}_k\,\mathrm{d}\widehat{\mathbf{x}} \tag{8}$$

and the interface divergence theorem

$$\int_{\mathcal{S}} \frac{\partial(\cdot)}{\partial x_l}\,\widehat{i}_{kl}\,\mathrm{d}\widehat{\mathbf{x}} = \int_{\partial\mathcal{S}} (\cdot)\widehat{m}_k\,\mathrm{d}\widehat{\mathbf{x}} + \int_{\mathcal{S}} \frac{\partial\widehat{n}_p}{\partial x_q}\,\widehat{i}_{pq}(\cdot)\widehat{n}_k\,\mathrm{d}\widehat{\mathbf{x}}, \tag{9}$$

together with the fact that the interface electric displacement $\widehat{\mathbf{d}}$ is a tangential vector — so that $\widehat{\mathbf{d}}\cdot\widehat{\mathbf{n}} = 0$ as a consequence of (6) — Gauss's law (7) can be rewritten in the localized or differential form

$$\begin{cases} \operatorname{div}\mathbf{d} = q, & \mathbf{x}\in\Omega\setminus\Gamma \\ \widehat{\operatorname{div}}\,\widehat{\mathbf{d}} - [\![\mathbf{d}]\!]\cdot\widehat{\mathbf{n}} = \widehat{q}, & \mathbf{x}\in\Gamma \\ \operatorname{div}\mathbf{d} = 0, & \mathbf{x}\in\mathbb{R}^3\setminus\Omega \\ [\![\mathbf{d}]\!]\cdot\mathbf{n} = 0, & \mathbf{x}\in\partial\Omega \end{cases}. \tag{10}$$

In these expressions, div is the standard divergence operator in the bulk and $\widehat{\operatorname{div}}$ stands for the interface divergence operator, that is, in indicial notation

$$\operatorname{div}\mathbf{d} = \frac{\partial d_k}{\partial x_k} \qquad \text{and} \qquad \widehat{\operatorname{div}}\,\widehat{\mathbf{d}} = \frac{\partial d_k}{\partial x_l}\widehat{i}_{kl}.$$

Remark 4.1. Note that equations (10) make it explicit that Gauss's law applies in the entirety of space, and hence also outside the body.

Lagrangian localized form Much like its integral counterpart (7), Gauss's law (10) is in Eulerian (spatial) form. For computational purposes, we shall

find it more convenient to deal with it in its Lagrangian (material) form

$$\begin{cases} \operatorname{Div}\mathbf{D} = Q, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \widehat{\operatorname{Div}}\,\widehat{\mathbf{D}} - [\![\mathbf{D}]\!]\cdot\widehat{\mathbf{N}} = \widehat{Q}, & \mathbf{X} \in \Gamma_0 \\ \operatorname{Div}\mathbf{D} = 0, & \mathbf{X} \in \mathbb{R}^3 \setminus \Omega \\ [\![\mathbf{D}]\!]\cdot\mathbf{N} = 0, & \mathbf{X} \in \partial\Omega_0 \end{cases}, \tag{11}$$

where $Q = Jq$ is the *space charge* density per unit initial volume, $\mathbf{D} = J\mathbf{F}^{-1}\mathbf{d}$ is the *Lagrangian electric displacement*, while $\widehat{Q} = \widehat{J}\widehat{q}$ stands for the *interface charge* density per unit initial area, and $\widehat{\mathbf{D}} = \widehat{J}\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{d}}$ stands for the *Lagrangian interface electric displacement*.

The notation utilized in equations (11) for the bulk and interface divergence operators in the initial configuration is entirely analogous to that employed in (10) in the current configuration: $\operatorname{Div}\mathbf{D} = \operatorname{tr}\nabla\mathbf{D} = \nabla\mathbf{D}\cdot\mathbf{I}$ and $\widehat{\operatorname{Div}}\,\widehat{\mathbf{D}} = \operatorname{tr}\widehat{\nabla}\widehat{\mathbf{D}} = \nabla\widehat{\mathbf{D}}\cdot\widehat{\mathbf{I}}$. A derivation of (11) starting from (10) is provided in Appendix A.

4.3 Faraday's law

Integral form In the absence of magnetic fields, electric currents, and time dependence, when Ampère's law and Gauss's law for magnetism are trivially satisfied, the integral form of Faraday's law reads

$$\int_{\partial\Sigma} \mathbf{e}\cdot\mathrm{d}\mathbf{x} = 0, \tag{12}$$

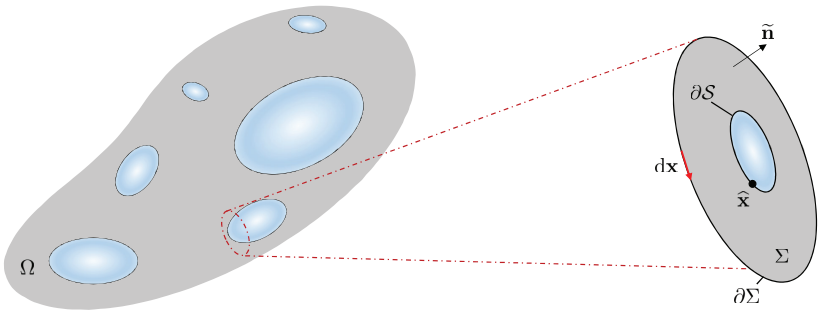

Figure 3. Schematic of an open surface Σ, cutting through a liquid inclusion, in the current configuration Ω indicating its unit normal $\tilde{\mathbf{n}}$ and boundary $\partial\Sigma$.

where Σ is any given open surface, with unit normal $\widetilde{\mathbf{n}}$, in the current configuration Ω and $\partial\Sigma$ denotes its boundary, a closed curve oriented in the usual sense with respect to $\widetilde{\mathbf{n}}$; see Fig. 3.

Localized form Making use of Stokes's theorem

$$\int_{\Sigma} (\operatorname{curl} \mathbf{e}) \cdot \widetilde{\mathbf{n}} \, \mathrm{d}\mathbf{x} = \int_{\partial\Sigma} \mathbf{e} \cdot \mathrm{d}\mathbf{x} + \int_{\partial\mathcal{S}} [\![\mathbf{e}]\!] \cdot \mathrm{d}\mathbf{x},$$

Faraday's law (12) can be rewritten in the localized form

$$\begin{cases} \operatorname{curl} \mathbf{e} = \mathbf{0}, & \mathbf{x} \in \Omega \setminus \Gamma \\ \widehat{\mathbf{i}} \, [\![\mathbf{e}]\!] = \mathbf{0}, & \mathbf{x} \in \Gamma \\ \operatorname{curl} \mathbf{e} = \mathbf{0}, & \mathbf{x} \in \mathbb{R}^3 \setminus \Omega \\ (\mathbf{I} - \mathbf{n} \otimes \mathbf{n}) \, [\![\mathbf{e}]\!] = \mathbf{0}, & \mathbf{x} \in \partial\Omega \end{cases} . \tag{13}$$

In these expressions, curl is the standard curl operator in the bulk, that is, in indicial notation

$$(\operatorname{curl} \mathbf{e})_i = \varepsilon_{ijk} \frac{\partial e_k}{\partial x_j}.$$

Remark 4.2. Analogous to (10), equations (13) make it explicit that Faraday's law applies in the entirety of space, and hence also outside the body.

Remark 4.3. Equations (13) imply that the electric field $\mathbf{e}$ is the gradient of an electric potential, say $\phi(\mathbf{x})$, that is a continuous function of $\mathbf{x}$. Precisely,

$$\begin{cases} \mathbf{e} = \nabla_{\mathbf{x}} \phi = \dfrac{\partial \phi}{\partial \mathbf{x}}, & \mathbf{x} \in \Omega \setminus \Gamma \\ [\![\phi]\!] = 0, & \mathbf{x} \in \Gamma \\ \mathbf{e} = \nabla_{\mathbf{x}} \phi = \dfrac{\partial \phi}{\partial \mathbf{x}}, & \mathbf{x} \in \mathbb{R}^3 \setminus \Omega \\ [\![\phi]\!] = 0, & \mathbf{x} \in \partial\Omega \end{cases} .$$

Lagrangian localized form For computational purposes, as already noted for Gauss's law above, we shall find it more convenient not to deal with the Eulerian form (13) of Faraday's law, but to do so with its Lagrangian form.

This reads

$$\begin{cases} \operatorname{Curl}\mathbf{E} = \mathbf{0}, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \widehat{\mathbf{I}}\,[\![\mathbf{E}]\!] = \mathbf{0}, & \mathbf{X} \in \Gamma_0 \\ \operatorname{Curl}\mathbf{E} = \mathbf{0}, & \mathbf{X} \in \mathbb{R}^3 \setminus \Omega_0 \\ (\mathbf{I} - \mathbf{N}\otimes\mathbf{N})\,[\![\mathbf{E}]\!] = \mathbf{0}, & \mathbf{X} \in \partial\Omega_0 \end{cases}, \tag{14}$$

where $\mathbf{E} = \mathbf{F}^T\mathbf{e}$ is the *Lagrangian electric field* and the curl operator is entirely analogous to that employed in (13) in the current configuration: $\operatorname{Curl}\mathbf{E} = \nabla \wedge \mathbf{E}$. A derivation of (14) starting from (12) is provided in Appendix B.

Remark 4.4. Of course, exactly like equations (13), equations (14) imply that the Lagrangian electric field $\mathbf{E}$ is the gradient of an electric potential, say $\Phi(\mathbf{X})$, that is a continuous function of $\mathbf{X}$. Precisely,

$$\begin{cases} \mathbf{E} = \nabla\Phi, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ [\![\Phi]\!] = 0, & \mathbf{X} \in \Gamma_0 \\ \mathbf{E} = \nabla\Phi, & \mathbf{X} \in \mathbb{R}^3 \setminus \Omega_0 \\ [\![\Phi]\!] = 0, & \mathbf{X} \in \partial\Omega_0 \end{cases}.$$

Clearly, $\Phi(\mathbf{X}) = \phi(\mathbf{y}(\mathbf{X}))$.

5 Balance of momenta in the presence of material interfaces

5.1 Bulk and interface electric and mechanical forces

In any given subdomain of the current configuration $\mathcal{D} \subset \Omega$, with boundary $\partial\mathcal{D}$ and outward unit normal $\widetilde{\mathbf{n}}$, the presence of the space charge density q, electric field $\mathbf{e}$, and electric displacement $\mathbf{d}$ described in the preceding section generates an *electric surface force* per unit current area, or electric surface traction, given by

$$\mathbf{t}_{\mathbf{e}}(\mathbf{x}) = \underbrace{\left[\mathbf{e}\otimes\mathbf{d} - \frac{1}{2}\varepsilon_0(\mathbf{e}\cdot\mathbf{e})\mathbf{I}\right]}_{\mathbf{T}_{\mathbf{e}}}\widetilde{\mathbf{n}}, \quad \mathbf{x} \in \partial\mathcal{D}, \tag{15}$$

where ε_0 is the permittivity of vacuum and $\mathbf{T}_{\mathbf{e}}$ is the so-called Maxwell stress.

Remark 5.1. Making use of the bulk divergence theorem (8), the electric surface traction (15) implies the presence of an *electric body force* per unit current volume, which reads

$$\mathbf{b}_{\mathsf{e}}(\mathbf{x}) = \operatorname{div} \mathbf{T}_{\mathsf{e}} = q\mathbf{e} + (\nabla_{\mathbf{x}}\mathbf{e})\,(\mathbf{d} - \varepsilon_0\mathbf{e})\,, \quad \mathbf{x} \in \mathcal{D};$$

see, e.g., Eq. (7.38) in the review by Pao (1978).

Remark 5.2. In direct analogy to (15), the interface charge density $\widehat{q}$ and interface electric displacement $\widehat{\mathbf{d}}$ described in the preceding section generate an *electric interface force* per unit current length, or electric interface traction, say $\widehat{\mathbf{t}}_{\mathsf{e}}(\widehat{\mathbf{x}})$, $\widehat{\mathbf{x}} \in \partial\mathcal{S}$. In this work, we take such an electric interface force to be negligible compared to the rest of forces.

In addition to the electric surface force (15), we consider that there may be three different types of mechanical forces present in any given subdomain $\mathcal{D} \subset \Omega$, to wit, a *mechanical body force* per unit current volume

$$\mathbf{b}_{\mathsf{m}}(\mathbf{x}), \quad \mathbf{x} \in \mathcal{D},$$

a *mechanical surface force* per unit current area, or mechanical surface traction,

$$\mathbf{t}_{\mathsf{m}}(\mathbf{x}, \widetilde{\mathbf{n}}), \quad \mathbf{x} \in \partial\mathcal{D},$$

and a *mechanical interface force* per unit current length, or mechanical interface traction,

$$\widehat{\mathbf{t}}_{\mathsf{m}}(\widehat{\mathbf{x}}, \widehat{\mathbf{m}}), \quad \widehat{\mathbf{x}} \in \partial\mathcal{S}.$$

Recall that $\partial\mathcal{S}$ stands for the boundary of any subsurfaces of the interfaces $\mathcal{S} \subset \Gamma$ that the subdomain $\mathcal{D}$ may contain. The first two of these three mechanical forces are standard. The third one accounts for the possibility of additional forces at the matrix/inclusions interfaces, such as, for instance, surface tension and Marangoni forces; see, e.g., Popinet (2018) and references therein. Within such a class of interface forces, in analogy to (6), we shall restrict attention to *tangential* forces in the sense that

$$\widehat{\mathbf{i}}\,\widehat{\mathbf{t}}_{\mathsf{m}} = \widehat{\mathbf{t}}_{\mathsf{m}}. \tag{16}$$

Note that Cauchy's fundamental postulate has been tacitly assumed to apply, thus the dependencies of the surface traction $\mathbf{t}_{\mathsf{m}}$ on $\widehat{\mathbf{n}}$ and of the interfacial traction $\widehat{\mathbf{t}}_{\mathsf{m}}$ on $\widehat{\mathbf{m}}$, which, again, stands for the outward unit normal to $\partial\mathcal{S}$. Figure 4 shows a schematic of a generic subdomain $\mathcal{D}$ with all four types of forces depicted.

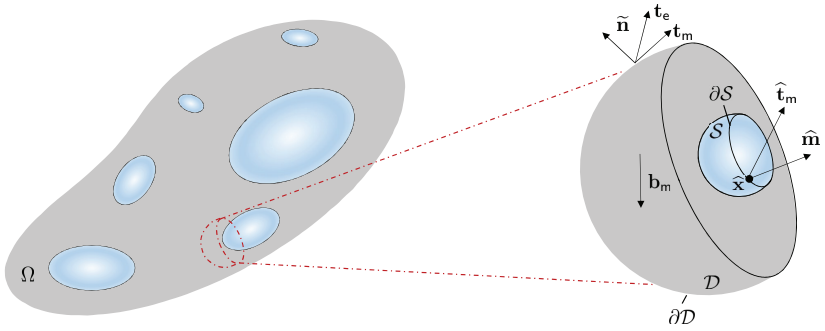

Figure 4. Schematic of a subdomain of the current configuration $\mathcal{D} \subset \Omega$, with boundary $\partial\mathcal{D}$ and outward unit normal $\widetilde{\mathbf{n}}$, indicating the electric surface force $\mathbf{t}_\mathsf{e}(\mathbf{x})$, mechanical body force $\mathbf{b}_\mathsf{m}(\mathbf{x})$, mechanical surface force $\mathbf{t}_\mathsf{m}(\mathbf{x}, \widetilde{\mathbf{n}})$, and mechanical interface force $\widehat{\mathbf{t}}_\mathsf{m}(\widehat{\mathbf{x}}, \widehat{\mathbf{m}})$ that is subjected to.

5.2 Balance of linear momentum

Integral form Absent inertia, granted the above-described types of electric and mechanical forces, balance of linear momentum reads

$$\int_{\partial\mathcal{D}} \mathbf{t}_\mathsf{e}\,(\mathbf{x})\ \mathrm{d}\mathbf{x} + \int_{\mathcal{D}} \mathbf{b}_\mathsf{m}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{D}} \mathbf{t}_\mathsf{m}\,(\mathbf{x}, \widetilde{\mathbf{n}})\ \mathrm{d}\mathbf{x} + \int_{\partial\mathcal{S}} \widehat{\mathbf{t}}_\mathsf{m}\,(\widehat{\mathbf{x}}, \widehat{\mathbf{m}})\ \mathrm{d}\widehat{\mathbf{x}} = \mathbf{0}. \tag{17}$$

Assuming that $\mathbf{t}_\mathsf{m}$ and $\widehat{\mathbf{t}}_\mathsf{m}$ are continuous in $\partial\mathcal{D} \setminus \partial\mathcal{S}$ and $\partial\mathcal{S}$, respectively, if follows from (17) that

$$\mathbf{t}_\mathsf{m}(\mathbf{x}, \widetilde{\mathbf{n}}) = \mathbf{T}_\mathsf{m}(\mathbf{x})\widetilde{\mathbf{n}}, \quad \mathbf{x} \in \mathcal{D} \setminus \mathcal{S}, \qquad \widehat{\mathbf{t}}_\mathsf{m}(\widehat{\mathbf{x}}, \widehat{\mathbf{m}}) = \widehat{\mathbf{T}}_\mathsf{m}(\widehat{\mathbf{x}})\widehat{\mathbf{m}}, \quad \widehat{\mathbf{x}} \in \mathcal{S}, \tag{18}$$

where $\mathbf{T}_\mathsf{m}$ is the standard Cauchy stress tensor in the bulk (resulting from mechanical forces) while $\widehat{\mathbf{T}}_\mathsf{m}$ is the *interface Cauchy stress* tensor. The former is continuous in $\mathcal{D} \setminus \mathcal{S}$ but may have a jump at $\mathcal{S}$, while the latter is continuous on $\mathcal{S}$ and, by virtue of (16), is a *tangential* tensor in the sense that $\widehat{\mathbf{i}}\widehat{\mathbf{T}}_\mathsf{m}\widehat{\mathbf{i}} = \widehat{\mathbf{T}}_\mathsf{m}$.

Localized form Making use of relations (18), the bulk divergence theorem (8), the interface divergence theorem (9), and the fact that $\widehat{\mathbf{T}}$ is a *superficial* tensor in the sense that $\widehat{\mathbf{T}}_\mathsf{m}\widehat{\mathbf{i}} = \widehat{\mathbf{T}}_\mathsf{m}$, the integral form (17) of the

balance of linear momentum can be rewritten as

$$\int_{\mathcal{D}} \operatorname{div} \mathbf{T}_{\mathsf{e}} \, \mathrm{d}\mathbf{x} - \int_{\mathcal{S}} [\![\mathbf{T}_{\mathsf{e}}]\!] \hat{\mathbf{n}} \, \mathrm{d}\hat{\mathbf{x}} + \int_{\mathcal{D}} \mathbf{b}_{\mathsf{m}}(\mathbf{x}) \, \mathrm{d}\mathbf{x} + \int_{\mathcal{D}} \operatorname{div} \mathbf{T}_{\mathsf{m}} \, \mathrm{d}\mathbf{x} - \int_{\mathcal{S}} [\![\mathbf{T}_{\mathsf{m}}]\!] \hat{\mathbf{n}} \, \mathrm{d}\hat{\mathbf{x}} + \int_{\mathcal{S}} \widehat{\operatorname{div}} \, \widehat{\mathbf{T}}_{\mathsf{m}} \, \mathrm{d}\hat{\mathbf{x}} = \mathbf{0},$$

from which one can readily determine the localized form

$$\begin{cases} \operatorname{div} \mathbf{T} + \mathbf{b} = \mathbf{0}, & \mathbf{x} \in \Omega \setminus \Gamma \\ \widehat{\operatorname{div}} \, \widehat{\mathbf{T}} - [\![\mathbf{T}]\!] \hat{\mathbf{n}} = \mathbf{0}, & \mathbf{x} \in \Gamma \end{cases} \tag{19}$$

in terms of the *total Cauchy stress tensor*

$$\mathbf{T} = \mathbf{T}_{\mathsf{m}} + \mathbf{T}_{\mathsf{e}} \tag{20}$$

in the bulk.

In these last expressions, for notational simplicity, we have dropped the subscript "m" in

$$\mathbf{b}_{\mathsf{m}} \mapsto \mathbf{b} \qquad \text{and} \qquad \widehat{\mathbf{T}}_{\mathsf{m}} \mapsto \widehat{\mathbf{T}}$$

since there is no longer risk of confusion. We also recall that div is the standard divergence operator in the bulk, while $\widehat{\operatorname{div}}$ stands for the interface divergence operator, namely, in indicial notation

$$(\operatorname{div} \mathbf{T})_i = \frac{\partial T_{ij}}{\partial x_j} \qquad \text{and} \qquad \left(\widehat{\operatorname{div}} \, \widehat{\mathbf{T}}\right)_i = \frac{\partial \widehat{T}_{ij}}{\partial x_k} \, \widehat{\imath}_{jk}.$$

Lagrangian localized form For computational purposes, once more, as already noted for the Maxwell's equations above, we favor dealing with the Lagrangian form of the equations of balance of linear momentum. Those read

$$\begin{cases} \operatorname{Div} \mathbf{S} + \mathbf{B} = \mathbf{0}, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \widehat{\operatorname{Div}} \, \widehat{\mathbf{S}} - [\![\mathbf{S}]\!] \widehat{\mathbf{N}} = \mathbf{0}, & \mathbf{X} \in \Gamma_0 \end{cases}, \tag{21}$$

where $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ is the *total first Piola-Kirchhoff stress tensor* in the bulk, $\mathbf{B} = J\mathbf{b}$ is the *mechanical body force* per unit initial volume, and $\widehat{\mathbf{S}} = \widehat{J}\widehat{\mathbf{T}}\widehat{\mathbf{F}}^{-T}$ stands for the *interface first Piola-Kirchhoff stress* tensor.

The notation utilized in (21) for the bulk divergence and interface divergence operators in the initial configuration is entirely analogous to that employed in (19) in the current configuration: $\operatorname{Div} \mathbf{S} = \nabla \mathbf{S} \cdot \mathbf{I}$ and $\widehat{\operatorname{Div}} \, \widehat{\mathbf{S}} = \nabla \widehat{\mathbf{S}} \cdot \widehat{\mathbf{I}}$. A derivation of (21) starting from (19) follows *mutatis mutandis* from the derivation given in Appendix A of (Ghosh and Lopez-Pamies (2022)) for the case when only mechanical forces are present.

Remark 5.3. It follows from the connection $\widehat{\mathbf{F}}^{-T} = \widehat{\mathbf{i}}\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{I}}$ that the interface first Piola-Kirchhoff stress $\widehat{\mathbf{S}}$ is a *superficial* tensor in the sense that $\widehat{\mathbf{S}}\widehat{\mathbf{I}} = \widehat{\mathbf{S}}$. Contrary to $\widehat{\mathbf{T}}$, however, $\widehat{\mathbf{S}}$ is *not* a *tangential* tensor since, in general, $\widehat{\mathbf{I}}\widehat{\mathbf{S}}\widehat{\mathbf{I}} \neq \widehat{\mathbf{S}}$.

5.3 Balance of angular momentum

Integral form In turn, absent inertia and granted the above-described types of electric and mechanical forces, balance of angular momentum reads

$$\int_{\partial\mathcal{D}} \mathbf{x} \wedge \mathbf{t}_{\mathsf{e}}\,(\mathbf{x})\ \mathrm{d}\mathbf{x} + \int_{\mathcal{D}} \mathbf{x} \wedge \mathbf{b}_{\mathsf{m}}(\mathbf{x})\,\mathrm{d}\mathbf{x} + \int_{\partial\mathcal{D}} \mathbf{x} \wedge \mathbf{t}_{\mathsf{m}}\,(\mathbf{x}, \widetilde{\mathbf{n}})\ \mathrm{d}\mathbf{x} + \int_{\partial\mathcal{S}} \mathbf{x} \wedge \widehat{\mathbf{t}}_{\mathsf{m}}\,(\widehat{\mathbf{x}}, \widehat{\mathbf{m}})\ \mathrm{d}\widehat{\mathbf{x}} = \mathbf{0}. \quad (22)$$

Localized form A standard calculation (see, e.g., Section 3.3.2 in the monograph by Ogden (1997)) shows that the integral form (22) of the balance of angular momentum can be written in the simple localized form

$$\begin{cases} \mathbf{T}^T = \mathbf{T}, & \mathbf{x} \in \Omega \setminus \Gamma \\ \widehat{\mathbf{T}}^T = \widehat{\mathbf{T}}, & \mathbf{x} \in \Gamma \end{cases} \quad (23)$$

in terms of the total Cauchy stress tensor (20) in the bulk and the interface Cauchy stress tensor $\widehat{\mathbf{T}}$, where, for notational simplicity, we have again dropped the subscript "m": $\widehat{\mathbf{T}}_{\mathsf{m}} \mapsto \widehat{\mathbf{T}}$.

Lagrangian localized form Given the definitions $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-T}$ and $\widehat{\mathbf{S}} = \widehat{J}\widehat{\mathbf{T}}\widehat{\mathbf{F}}^{-T}$ of the total bulk and interface first Piola-Kirchhoff stress tensors, it is a simple matter to deduce from the balance of angular momentum (23) in Eulerian form that the balance of angular momentum in Lagrangian form is given by

$$\begin{cases} \mathbf{S}\mathbf{F}^T = \mathbf{F}\mathbf{S}^T, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \widehat{\mathbf{S}}\widehat{\mathbf{F}}^T = \widehat{\mathbf{F}}\widehat{\mathbf{S}}^T, & \mathbf{X} \in \Gamma_0 \end{cases} . \quad (24)$$

6 Constitutive behavior

For a given initial configuration Ω_0 of the body, given initial mass density ρ_0, given bulk and interface charges Q and $\widehat{Q}$, and given mechanical body force $\mathbf{B}$, mass conservation (5), the Maxwell's equations (11) and (14), and the

balance of linear and angular momenta (21) and (24) are *coupled* equations for the deformation field $\mathbf{y}$, the mass density ρ, the Lagrangian electric displacement $\mathbf{D}$, the Lagrangian electric field $\mathbf{E}$, the Lagrangian interface electric displacement $\widehat{\mathbf{D}}$, the total first Piola-Kirchhoff stress tensor $\mathbf{S}$, and the interface first Piola-Kirchhoff stress tensor $\widehat{\mathbf{S}}$ that apply *generally*.

The next step in the formulation of a mathematically closed system of governing equations is to describe the intrinsic electromechanical properties of the materials that the body is made of, precisely, the constitutive behavior of: the solid that the matrix occupying the subdomain $\Omega_0^{\mathtt{m}}$ is made of, the liquid that the inclusions occupying the subdomain $\Omega_0^{\mathtt{i}}$ are made of, and the solid/liquid interfaces Γ_0.

6.1 Constitutive behavior of the bulk: The solid matrix and the liquid inclusions

Constitutive behavior of the solid matrix The focus of this work is on material systems wherein the underlying solid matrix is an elastomer. Accordingly, neglecting dissipative phenomena, we model the matrix as an elastic dielectric solid. Precisely, making use of the formulation introduced by Dorfmann and Ogden (2005), we find it convenient to characterize the electromechanical behavior of the solid matrix in a Lagrangian formulation by a total free energy (per unit initial volume)

$$W_{\mathtt{m}} = W_{\mathtt{m}}(\mathbf{F}, \mathbf{E}),$$

that is an objective function of the deformation gradient tensor $\mathbf{F}$ and an objective and even function of the Lagrangian electric field $\mathbf{E}$, so that

$$W_{\mathtt{m}}(\mathbf{QF}, \mathbf{E}) = W_{\mathtt{m}}(\mathbf{F}, \mathbf{E}) \quad \text{and} \quad W_{\mathtt{m}}(\mathbf{F}, -\mathbf{E}) = W_{\mathtt{m}}(\mathbf{F}, \mathbf{E}) \quad \forall\, \mathbf{Q} \in SO(3)$$

and arbitrary $\mathbf{F}$ and $\mathbf{E}$.

In the sequel, for clarity of presentation, we will restrict attention to the basic case of an ideal elastic dielectric for which the free energy reads

$$W_{\mathtt{m}}(\mathbf{F}, \mathbf{E}) = \frac{\mu_{\mathtt{m}}}{2}\,[\,\mathbf{F}\cdot\mathbf{F} - 3] - \mu_{\mathtt{m}} \ln J + \frac{\Lambda_{\mathtt{m}}}{2}(J-1)^2 - \frac{\varepsilon_{\mathtt{m}}}{2} J\mathbf{F}^{-T}\mathbf{E}\cdot\mathbf{F}^{-T}\mathbf{E}. \quad (25)$$

In this constitutive prescription, the material constants $\mu_{\mathtt{m}} > 0$, $\Lambda_{\mathtt{m}} > 0$, $\varepsilon_{\mathtt{m}} \geq \varepsilon_0$ stand for the initial Lamé constants and the initial permittivity of the elastomer under consideration.

Remark 6.1. Here, it is important to emphasize that the use of a free energy of the form (25) implies that, in its initial configuration, the elastomeric matrix is stress and polarization free. In other words, we are assuming that

there are *no* residual stress and *no* residual polarization in the elastomeric matrix. Depending on the fabrication process of the filled elastomer of interest, however, this assumption may not be appropriate. As elaborated below in Section 7, this assumption is indeed appropriate for the prototypical case when the liquid inclusions are initially *spherical* in shape.

Constitutive behavior of the liquid inclusions Granted the absence of inertia, the liquid making up the inclusions is presumed to behave as an elastic dielectric fluid. For clarity of presentation, we consider in particular that the electromechanical behavior of the liquid inclusions is characterized by the free energy

$$W_{\mathtt{i}}^{j}(\mathbf{X},\mathbf{F},\mathbf{E}) = r_{\mathtt{i}}^{j}(\mathbf{X})J + \frac{\Lambda_{\mathtt{i}}}{2}(J-1)^2 - \frac{\varepsilon_{\mathtt{i}}}{2}J\mathbf{F}^{-T}\mathbf{E}\cdot\mathbf{F}^{-T}\mathbf{E} \quad j=1,2,...,M, \tag{26}$$

where, as will become apparent below in Section 7, $r_{\mathtt{i}}^{j}(\mathbf{X})$ shall stand for the pressure — which is *not* necessarily zero due to the possible presence of initial interfacial forces — that the liquid within the jth inclusion is subjected to in the initial configuration, when $\mathbf{F}=\mathbf{I}$ and $\mathbf{E}=\mathbf{0}$, while $\Lambda_{\mathtt{i}} \geq 0$ and $\varepsilon_{\mathtt{i}} \geq \varepsilon_0$ denote the initial first Lamé constant (or bulk modulus since $\mu_{\mathtt{i}}=0$) and the initial permittivity of the liquid, respectively.

Remark 6.2. Given the constitutive prescription (26), the case of an incompressible liquid corresponds to setting $\Lambda_{\mathtt{i}} = +\infty$, while the case of a conducting liquid corresponds to setting $\varepsilon_{\mathtt{i}} = +\infty$.

Remark 6.3. All M inclusions are assumed to be made of the same liquid, thus the unique values of $\Lambda_{\mathtt{i}}$ and $\varepsilon_{\mathtt{i}}$ in (26). However, because each inclusion is allowed to have its own initial geometry, and thus its own initial size, the term $r_{\mathtt{i}}^{j}(\mathbf{X})$ in (26) describing the residual stress within the inclusions may be different for each inclusion.

Pointwise constitutive behavior of the bulk Given the indicator functions (1) for the inclusions and the free energies (25) and (26) for the matrix and the inclusions, the pointwise free energy for the bulk of the body can be compactly written as

$$\begin{aligned} W(\mathbf{X},\mathbf{F},\mathbf{E}) =& r_{\mathtt{i}}(\mathbf{X})J + \\ & \frac{\mu(\mathbf{X})}{2}\left[\mathbf{F}\cdot\mathbf{F}-3\right] - \mu(\mathbf{X})\ln J + \frac{\Lambda(\mathbf{X})}{2}\left(J-1\right)^2 - \\ & \frac{\varepsilon(\mathbf{X})}{2}J\mathbf{F}^{-T}\mathbf{E}\cdot\mathbf{F}^{-T}\mathbf{E}, \end{aligned} \tag{27}$$

where

$$\begin{cases} r_{\mathtt{i}}(\mathbf{X}) = \displaystyle\sum_{j=1}^{M} \theta_0^{\mathtt{i},j}(\mathbf{X}) r_{\mathtt{i}}^j(\mathbf{X}) \\ \mu(\mathbf{X}) = (1-\theta_0^{\mathtt{i}}(\mathbf{X}))\,\mu_{\mathtt{m}} \\ \Lambda(\mathbf{X}) = (1-\theta_0^{\mathtt{i}}(\mathbf{X}))\,\Lambda_{\mathtt{m}} + \theta_0^{\mathtt{i}}(\mathbf{X})\Lambda_{\mathtt{i}} \\ \varepsilon(\mathbf{X}) = (1-\theta_0^{\mathtt{i}}(\mathbf{X}))\,\varepsilon_{\mathtt{m}} + \theta_0^{\mathtt{i}}(\mathbf{X})\varepsilon_{\mathtt{i}} \end{cases} .$$

It then follows that the total first Piola-Kirchhoff stress tensor $\mathbf{S}$ and the Lagrangian electric displacement $\mathbf{D}$ at any material point in the bulk are given by the relations

$$\begin{aligned} \mathbf{S}(\mathbf{X}) = \frac{\partial W}{\partial \mathbf{F}}(\mathbf{X},\mathbf{F},\mathbf{E}) =& r_{\mathtt{i}}(\mathbf{X}) J\mathbf{F}^{-T} + \\ & \mu(\mathbf{X})\left(\mathbf{F}-\mathbf{F}^{-T}\right) + \Lambda(\mathbf{X})(J-1)J\mathbf{F}^{-T} + \\ & \varepsilon(\mathbf{X})J\mathbf{F}^{-T}\mathbf{E}\otimes\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{E} - \\ & \frac{\varepsilon(\mathbf{X})}{2}(\mathbf{F}^{-T}\mathbf{E}\cdot\mathbf{F}^{-T}\mathbf{E})J\mathbf{F}^{-T}, \quad \mathbf{X}\in\Omega_0\setminus\Gamma_0 \end{aligned} \tag{28}$$

and

$$\mathbf{D}(\mathbf{X}) = -\frac{\partial W}{\partial \mathbf{E}}(\mathbf{X},\mathbf{F},\mathbf{E}) = \varepsilon(\mathbf{X})J\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{E}, \quad \mathbf{X}\in\Omega_0\setminus\Gamma_0. \tag{29}$$

Remark 6.4. In the limit of small deformations and moderate electric fields[2] — when $\mathbf{H}=\mathbf{F}-\mathbf{I}$ is of $O(\zeta)$, $\mathbf{E}$ is of $O(\zeta^{1/2})$, and $\zeta \searrow 0$ — the coupled constitutive response (28)-(29) reduces asymptotically to

$$\begin{aligned} \mathbf{S}(\mathbf{X}) =& r_{\mathtt{i}}(\mathbf{X})\mathbf{I} - r_{\mathtt{i}}(\mathbf{X})\mathbf{H}^T + r_{\mathtt{i}}(\mathbf{X})(\operatorname{tr}\mathbf{H})\mathbf{I} + \\ & \mu(\mathbf{X})\left(\mathbf{H}+\mathbf{H}^T\right) + \Lambda(\mathbf{X})(\operatorname{tr}\mathbf{H})\mathbf{I} + \\ & \varepsilon(\mathbf{X})\mathbf{E}\otimes\mathbf{E} - \frac{\varepsilon(\mathbf{X})}{2}(\mathbf{E}\cdot\mathbf{E})\,\mathbf{I} + O(\zeta^2) \end{aligned} \tag{30}$$

and

$$\mathbf{D}(\mathbf{X}) = \varepsilon(\mathbf{X})\mathbf{E} + O(\zeta^{3/2}). \tag{31}$$

[2] For studies of this fundamental limit, see, e.g., Stratton (1941); Tian et al. (2012); Lefèvre and Lopez-Pamies (2014); Spinelli et al. (2015).

The corresponding total Cauchy stress tensor $\mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T$ and Eulerian electric displacement $\mathbf{d} = J^{-1}\mathbf{F}\mathbf{D}$ are given by

$$\begin{aligned}\mathbf{T}(\mathbf{x}) =& r_{\mathtt{i}}(\mathbf{x})\mathbf{I}+\\ &\mu(\mathbf{x})\left(\mathbf{H}+\mathbf{H}^T\right)+\Lambda(\mathbf{x})(\operatorname{tr}\mathbf{H})\mathbf{I}+\\ &\varepsilon(\mathbf{x})\mathbf{E}\otimes\mathbf{E}-\frac{\varepsilon(\mathbf{x})}{2}\left(\mathbf{E}\cdot\mathbf{E}\right)\mathbf{I}+O(\zeta^2)\end{aligned} \tag{32}$$

and

$$\mathbf{d}(\mathbf{x}) = \varepsilon(\mathbf{x})\mathbf{E}+O(\zeta^{3/2}). \tag{33}$$

Three key features are now immediate. First, in the initial configuration, when $\mathbf{x}=\mathbf{X}$, $\mathbf{F}=\mathbf{I}$, and $\mathbf{E}=\mathbf{0}$, the Lagrangian relations (30)-(31) and the Eulerian relations (32)-(33) reduce to

$$\left\{\begin{array}{l}\mathbf{S}(\mathbf{X}) = r_{\mathtt{i}}(\mathbf{X})\mathbf{I}\\ \mathbf{D}(\mathbf{X}) = \mathbf{0}\end{array}\right. \quad \text{and} \quad \left\{\begin{array}{l}\mathbf{T}(\mathbf{x}) = r_{\mathtt{i}}(\mathbf{x})\mathbf{I}\\ \mathbf{d}(\mathbf{x}) = \mathbf{0}\end{array}\right. ,$$

which indicate that the inclusions (but *not* the matrix) have a hydrostatic residual stress. They also indicate that there is no residual polarization. Second, the stress (30) is *not* symmetric as it does not depend only on the symmetric part of $\mathbf{H}$, but also on $\mathbf{H}$ itself. Third, the total first Piola-Kirchhoff stress (30) does *not* coincide with the total Cauchy stress (32) to $O(||\mathbf{H}||)$. As discussed at length by Ghosh and Lopez-Pamies (2022) and also as elaborated below, these three non-standard features are direct consequences of the presence of a residual stress, which in turn is a direct consequence of the presence of interfacial forces.

Remark 6.5. Thanks to the objectivity of the free energies (25) and (26), the constitutive relation (28) satisfies automatically the balance of angular momentum $(24)_1$ in the bulk.

6.2 Constitutive behavior of the solid/liquid interfaces

Next, we turn to the constitutive description of the interfaces. Similar to the elastomeric matrix and liquid inclusions, we also consider that under the quasistatic deformations and quasistatic electric fields of interest here any (electric or mechanical) interfacial dissipative phenomena is negligible and hence presume the interfaces to exhibit an elastic dielectric behavior. Specifically, we consider that the interface first Piola-Kirchhoff stress tensor $\widehat{\mathbf{S}}$ and the Lagrangian interface electric displacement $\widehat{\mathbf{D}}$ are given by

relations of the form

$$\widehat{\mathbf{S}}(\mathbf{X}) = \frac{\partial \widehat{W}}{\partial \widehat{\mathbf{F}}}(\widehat{\mathbf{F}}, \widehat{\mathbf{E}}), \quad \mathbf{X} \in \Gamma_0 \tag{34}$$

and

$$\widehat{\mathbf{D}}(\mathbf{X}) = -\frac{\partial \widehat{W}}{\partial \widehat{\mathbf{E}}}(\widehat{\mathbf{F}}, \widehat{\mathbf{E}}), \quad \mathbf{X} \in \Gamma_0 \tag{35}$$

in terms of a suitably well-behaved interface free energy (per unit initial area) $\widehat{W}(\widehat{\mathbf{F}}, \widehat{\mathbf{E}})$, where we recall that $\widehat{\mathbf{F}}$ stands for the interface deformation gradient (2) and

$$\widehat{\mathbf{E}}(\mathbf{X}) = \widehat{\mathbf{I}}\mathbf{E}(\mathbf{X})$$

is the *Lagrangian interface electric field*.

In the sequel, for clarity of presentation, we will restrict attention to the ideal-elastic-dielectric-type interface free energy

$$\widehat{W}(\widehat{\mathbf{F}}, \widehat{\mathbf{E}}) = \widehat{\gamma}\, \widehat{J} + \frac{\widehat{\mu}}{2}\left[\widehat{\mathbf{F}} \cdot \widehat{\mathbf{F}} - 2\right] - \widehat{\mu} \ln \widehat{J} + \frac{\widehat{\Lambda}}{2}(\widehat{J} - 1)^2 - \frac{\widehat{\varepsilon}}{2}\widehat{J}\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} \cdot \widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}}. \tag{36}$$

In this constitutive prescription, the material constant $\widehat{\gamma} \geq 0$ describes the initial *surface tension* on the solid/liquid interfaces under consideration. On the other hand, $\widehat{\mu} \geq 0$, $\widehat{\Lambda} \geq 0$, and $\widehat{\varepsilon} \geq 0$ can be viewed as the initial *interface Lamé constants* and the initial *interface permittivity*. All three material constants $\widehat{\gamma}$, $\widehat{\mu}$, $\widehat{\Lambda}$ have units of $force/length$. On the other hand, the material constant $\widehat{\varepsilon}$ has units of $force \times length/voltage^2$.

A direct calculation shows that the interface first Piola-Kirchhoff stress tensor (34) and the Lagrangian interface electric displacement (35) associated with the interface free energy (36) are given by

$$\begin{aligned}\widehat{\mathbf{S}}(\mathbf{X}) = \frac{\partial \widehat{W}}{\partial \widehat{\mathbf{F}}}(\widehat{\mathbf{F}}, \widehat{\mathbf{E}}) =& \widehat{\gamma}\, \widehat{J}\widehat{\mathbf{F}}^{-T} + \\ & \widehat{\mu}(\widehat{\mathbf{F}} - \widehat{\mathbf{F}}^{-T}) + \widehat{\Lambda}(\widehat{J} - 1)\widehat{J}\widehat{\mathbf{F}}^{-T} + \\ & \widehat{\varepsilon}\widehat{J}\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} \otimes \widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} - \\ & \frac{\widehat{\varepsilon}}{2}(\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} \cdot \widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}})\widehat{J}\widehat{\mathbf{F}}^{-T}, \quad \mathbf{X} \in \Gamma_0\end{aligned} \tag{37}$$

and

$$\widehat{\mathbf{D}}(\mathbf{X}) = -\frac{\partial \widehat{W}}{\partial \widehat{\mathbf{E}}}(\widehat{\mathbf{F}}, \widehat{\mathbf{E}}) = \widehat{\varepsilon}\widehat{J}\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}}, \quad \mathbf{X} \in \Gamma_0. \tag{38}$$

Remark 6.6. Another direct calculation shows that the interface Cauchy stress tensor $\widehat{\mathbf{T}} = \widehat{J}^{-1}\widehat{\mathbf{S}}\widehat{\mathbf{F}}^T$ associated with the free energy (36) reads

$$\begin{aligned}\widehat{\mathbf{T}}(\mathbf{x}) =& \widehat{\gamma}\,\widehat{\mathbf{i}} + \widehat{\mu}(\widehat{J}^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}^T - \widehat{\mathbf{i}}) + \widehat{\Lambda}(\widehat{J}-1)\widehat{\mathbf{i}} + \\ & \widehat{\varepsilon}\,\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} \otimes \widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} - \frac{\widehat{\varepsilon}}{2}(\widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} \cdot \widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}})\widehat{\mathbf{i}}, \quad \mathbf{x} \in \Gamma. \end{aligned} \tag{39}$$

This expression makes it plain that the constitutive relation (37) utilized here to describe the electromechanical behavior of the interfaces generalizes in three counts the basic constitutive relation of constant surface-tension stress

$$\widehat{\mathbf{T}}(\mathbf{x}) = \widehat{\gamma}\widehat{\mathbf{i}}.$$

Specifically, the constitutive relation (39) includes Neo-Hookean-type deviatoric elasticity, via the term $\widehat{\mu}(\widehat{J}^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{F}}^T - \widehat{\mathbf{i}})$, and not just surface tension. It also accounts for a surface tension that is not necessarily a constant but instead one that depends on the deformation of the interface via the term $\widehat{\Lambda}(\widehat{J}-1)\widehat{\mathbf{i}}$. Finally, the last two terms in the constitutive relation (39) describe the presence of an interfacial polarization.

Remark 6.7. Thanks to the objectivity of the free energy (36), the constitutive relation (37) satisfies automatically the balance of angular momentum $(24)_2$ on the interfaces.

7 Governing equations

7.1 Boundary conditions

In terms of the external stimuli applied to the body, we have already described the source terms of bulk and interface charges Q and $\widehat{Q}$ and the mechanical body force $\mathbf{B}$. We now describe the external stimuli applied on the boundary of the body.

From an electric point of view, we take that the body is immersed in a surrounding space, e.g., air, where there is a heterogeneous electric field $\overline{\mathbf{E}}(\mathbf{X})$ and corresponding electric displacement $\overline{\mathbf{D}}(\mathbf{X})$ that result by the use of electrodes, where a surface charge density per unit initial area $\overline{Q}$ is applied, and/or the nearby presence of polarized bodies and by the interaction of these with the body. We then have the boundary condition

$$(\mathbf{I} - \mathbf{N} \otimes \mathbf{N})\mathbf{E} = (\mathbf{I} - \mathbf{N} \otimes \mathbf{N})\overline{\mathbf{E}}, \quad \mathbf{X} \in \partial\Omega_0, \tag{40}$$

or, equivalently,

$$\mathbf{D}\cdot\mathbf{N} = -\overline{Q} + \overline{\mathbf{D}}\cdot\mathbf{N}, \quad \mathbf{X}\in\partial\Omega_0,$$

over the entirety of the boundary of the domain occupied by the body.

From a mechanical point of view, on a portion $\partial\Omega_0^{\mathcal{D}}$ of the boundary $\partial\Omega_0$, the deformation field $\mathbf{y}$ is taken to be given by a known function $\overline{\mathbf{y}}(\mathbf{X})$, while the complementary part of the boundary $\partial\Omega_0^{\mathcal{N}} = \partial\Omega_0 \setminus \partial\Omega_0^{\mathcal{D}}$ is subjected to a prescribed mechanical traction $\overline{\mathbf{t}}_{\mathsf{m}}(\mathbf{X})$. Precisely,

$$\mathbf{y} = \overline{\mathbf{y}}, \quad \mathbf{X}\in\partial\Omega_0^{\mathcal{D}} \qquad \text{and} \qquad \mathbf{S}\mathbf{N} = \overline{\mathbf{t}}_{\mathsf{m}} + \overline{\mathbf{S}}_{\mathsf{e}}\mathbf{N}, \quad \mathbf{X}\in\partial\Omega_0^{\mathcal{N}}. \tag{41}$$

In this last expression, $\overline{\mathbf{S}}_{\mathsf{e}}$ stands for the Maxwell stress outside of the body. In the case when the body is surrounded by air,

$$\overline{\mathbf{S}}_{\mathsf{e}} = \mathbf{F}^{-T}\mathbf{E}\otimes\mathbf{D} - \frac{J\varepsilon_0}{2}\left(\mathbf{F}^{-T}\mathbf{E}\cdot\mathbf{F}^{-T}\mathbf{E}\right)\mathbf{F}^{-T},$$

where $\mathbf{D} = \varepsilon_0 J\mathbf{F}^{-1}\mathbf{F}^{-T}\mathbf{E}$ and where we remark that the deformation gradient $\mathbf{F}$ in the air refers to any suitably well-behaved extension of the deformation gradient $\mathbf{F}$ in the body.

7.2 The choice of independent fields

At this stage, all that remains to formulate a mathematically closed system of governing equations is to identify the independent fields that we wish to solve for. Arguably, the most expedient choice of independent fields for the problem at hand is the deformation field

$$\mathbf{y}(\mathbf{X})$$

and the electric potential

$$\Phi(\mathbf{X}).$$

Recall that the rest of fields can be written in terms of these two as follows:

$$\begin{cases} \mathbf{F} = \nabla\mathbf{y} \\ \widehat{\mathbf{F}} = \mathbf{F}\widehat{\mathbf{I}} \\ \mathbf{E} = \nabla\Phi \\ \widehat{\mathbf{E}} = \widehat{\mathbf{I}}\mathbf{E} \end{cases}, \qquad \begin{cases} \mathbf{S} = \dfrac{\partial W}{\partial\mathbf{F}}(\mathbf{X},\mathbf{F},\mathbf{E}) \\ \widehat{\mathbf{S}} = \dfrac{\partial\widehat{W}}{\partial\widehat{\mathbf{F}}}(\widehat{\mathbf{F}},\widehat{\mathbf{E}}) \\ \mathbf{D} = -\dfrac{\partial W}{\partial\mathbf{E}}(\mathbf{X},\mathbf{F},\mathbf{E}) \\ \widehat{\mathbf{D}} = -\dfrac{\partial\widehat{W}}{\partial\widehat{\mathbf{E}}}(\widehat{\mathbf{F}},\widehat{\mathbf{E}}) \end{cases}, \qquad \begin{cases} \rho = J^{-1}\rho_0 \\ \mathbf{T} = J^{-1}\mathbf{S}\mathbf{F}^T \\ \widehat{\mathbf{T}} = \widehat{J}^{-1}\widehat{\mathbf{S}}\widehat{\mathbf{F}}^T \\ \mathbf{d} = J^{-1}\mathbf{F}\mathbf{D} \\ \widehat{\mathbf{d}} = \widehat{J}^{-1}\widehat{\mathbf{F}}\widehat{\mathbf{D}} \\ \mathbf{e} = \mathbf{F}^{-T}\mathbf{E} \\ \widehat{\mathbf{e}} = \widehat{\mathbf{F}}^{-T}\widehat{\mathbf{E}} \\ \phi = \Phi \end{cases}.$$

7.3 The strong form of the governing equations

Granted the choice of the deformation field $\mathbf{y}(\mathbf{X})$ and the electric potential $\Phi(\mathbf{X})$ as the independent fields, the equation of conservation of mass (5) and Faraday's law (14) are automatically satisfied. Granted the use of objective free energies $W(\mathbf{X},\mathbf{F},\mathbf{E})$ and objective interface free energies $\widehat{W}(\widehat{\mathbf{F}},\widehat{\mathbf{E}})$, such as (27) and (36), the equations of balance of angular momentum (24) are also automatically satisfied. Thus, Gauss's law (11) and the balance of linear momentum (21) are the only two sets of balance principles that need to be solved.

Substitution of the constitutive relations (28), (29), (37) and (38) for the bulk and interfaces in Gauss's law (11), the balance of linear momentum (21), and the boundary conditions (40) and (41), together with use of the notation $\overline{\mathbf{E}} = \nabla\overline{\Phi}(\mathbf{X})$, yields the following governing equations

$$\begin{cases} \operatorname{Div}\left[-\dfrac{\partial W}{\partial \mathbf{E}}(\mathbf{X},\nabla\mathbf{y},\nabla\Phi)\right] = Q, \quad \mathbf{X}\in\Omega_0\setminus\Gamma_0 \\ \widehat{\operatorname{Div}}\left[-\dfrac{\partial \widehat{W}}{\partial \widehat{\mathbf{E}}}(\widehat{\nabla}\mathbf{y},\widehat{\nabla}\Phi)\right] - \left[\!\left[-\dfrac{\partial W}{\partial \mathbf{E}}(\mathbf{X},\nabla\mathbf{y},\nabla\Phi)\right]\!\right]\widehat{\mathbf{N}} = \widehat{Q}, \quad \mathbf{X}\in\Gamma_0 \\ \Phi(\mathbf{X}) = \overline{\Phi}(\mathbf{X}), \quad \mathbf{X}\in\partial\Omega_0 \end{cases} \tag{42}$$

and

$$\begin{cases} \operatorname{Div}\left[\dfrac{\partial W}{\partial \mathbf{F}}(\mathbf{X},\nabla\mathbf{y},\nabla\Phi)\right] + \mathbf{B} = \mathbf{0}, \quad \mathbf{X}\in\Omega_0\setminus\Gamma_0 \\ \widehat{\operatorname{Div}}\left[\dfrac{\partial \widehat{W}}{\partial \widehat{\mathbf{F}}}(\widehat{\nabla}\mathbf{y},\widehat{\nabla}\Phi)\right] - \left[\!\left[\dfrac{\partial W}{\partial \mathbf{F}}(\mathbf{X},\nabla\mathbf{y},\nabla\Phi)\right]\!\right]\widehat{\mathbf{N}} = \mathbf{0}, \quad \mathbf{X}\in\Gamma_0 \\ \mathbf{y}(\mathbf{X}) = \overline{\mathbf{y}}(\mathbf{X}), \quad \mathbf{X}\in\partial\Omega_0^{\mathcal{D}} \\ \left[\dfrac{\partial W}{\partial \mathbf{F}}(\mathbf{X},\nabla\mathbf{y},\nabla\Phi)\right]\mathbf{N} = \overline{\mathbf{t}}_{\mathrm{m}}(\mathbf{X}) + \overline{\mathbf{S}}_{\mathrm{e}}\mathbf{N}, \quad \mathbf{X}\in\partial\Omega_0^{\mathcal{N}} \end{cases} \tag{43}$$

for the deformation field $\mathbf{y}(\mathbf{X})$ and the electric potential $\Phi(\mathbf{X})$.

Equations (42)-(43) constitute a generalization of the classical governing equations for heterogenous elastic dielectrics under quasistatic deformations and quasistatic electric fields that accounts for: (i) the presence of residual stresses (in the inclusions) and (ii) jump conditions across material (solid/liquid) interfaces that are not algebraic but, instead, are described by PDEs (partial differential equations) that result from the presence of interfacial polarization and forces.

7.4 Residual stresses

In the initial configuration, prior to the application of the charges Q, $\widehat{Q}$, the body force $\mathbf{B}$, and the boundary conditions $\overline{\mathbf{y}}$, $\overline{\mathbf{t}}_{\mathsf{m}}$, and $\overline{\mathbf{S}}_{\mathsf{e}}$, the deformation field $\mathbf{y}(\mathbf{X}) = \mathbf{X}$ and the electric potential $\Phi(\mathbf{X}) = 0$ and hence the governing equations (42)-(43) reduce to

$$\left\{\begin{array}{l} \mathrm{Div}\,[r_{\mathtt{i}}(\mathbf{X})\mathbf{I}] = \mathbf{0}, \quad \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \\ \widehat{\gamma}\,\widehat{\mathrm{Div}}\,\widehat{\mathbf{I}} - [\![r_{\mathtt{i}}(\mathbf{X})]\!]\,\widehat{\mathbf{N}} = \mathbf{0}, \quad \mathbf{X} \in \Gamma_0 \end{array}\right. , \tag{44}$$

which can be viewed as the definition of the hydrostatic residual stress $r_{\mathtt{i}}(\mathbf{X})$ within the inclusions required to balance out the interfacial forces. Recognizing that $[\![r_{\mathtt{i}}(\mathbf{X})]\!] = r_{\mathtt{i}}(\mathbf{X})$ and that

$$\widehat{\mathrm{Div}}\,\widehat{\mathbf{I}} = -\nabla(\widehat{\mathbf{N}} \otimes \widehat{\mathbf{N}}) \cdot \widehat{\mathbf{I}} = -(\widehat{\mathbf{I}} \cdot \nabla\widehat{\mathbf{N}})\widehat{\mathbf{N}} = -(\mathrm{tr}\,\widehat{\nabla}\widehat{\mathbf{N}})\widehat{\mathbf{N}} = 2\kappa\widehat{\mathbf{N}}$$

in terms of the mean curvature $\kappa = -\mathrm{tr}\,\widehat{\nabla}\widehat{\mathbf{N}}/2$ of the interfaces, equations (44) can be rewritten more explicitly as

$$\left\{\begin{array}{l} \nabla r_{\mathtt{i}}(\mathbf{X}) = \mathbf{0}, \quad \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \\ r_{\mathtt{i}}(\mathbf{X}) = 2\kappa\widehat{\gamma}, \quad \mathbf{X} \in \Gamma_0 \end{array}\right. . \tag{45}$$

The PDE $(45)_1$ states that the hydrostatic residual stress $r_{\mathtt{i}}(\mathbf{X})$ must be *constant* — possibly a different constant — within each inclusion. In view of the boundary condition $(45)_2$, which is nothing more than the standard Young-Laplace equation, a solution to the boundary-value problem (45) then only exists for the case when all M inclusions have shapes of *constant mean curvature* (Kenmotsu (2003)), for only then $(45)_2$ is consistent with $(45)_1$. Physically, as alluded to in Remark 6.1, this result implies that to deal with liquid inclusions of general initial shape, one would have to account for residual stresses in the elastomeric matrix and not just within the inclusions.

Remark 7.1. The prototypical case of elastomers filled with liquid inclusions that have constant mean curvature — and hence for which the governing equations (42)-(43) apply — is that of elastomers that are filled with liquid inclusions that are initially *spherical* in shape. For these, the solution to (45) simply reads

$$r_{\mathtt{i}}(\mathbf{X}) = -\sum_{j=1}^{M} \theta_0^{\mathtt{i},j}(\mathbf{X})\frac{2\,\widehat{\gamma}}{A_j},$$

where A_j denotes the initial radius of the jth inclusion.

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Appendix A. Gauss's law in Lagrangian form

On substitution of the definitions $Q = Jq$, $\mathbf{D} = J\mathbf{F}^{-1}\mathbf{d}$, $\widehat{Q} = \widehat{J}\widehat{q}$, and $\widehat{\mathbf{D}} = \widehat{J}\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{d}}$ in the Eulerian form (10) of Gauss's law, we have

$$\begin{cases} \dfrac{\partial}{\partial x_k}\left[J^{-1}F_{km}D_m\right] = J^{-1}Q, & \mathbf{x}\in\Omega\setminus\Gamma \\ \dfrac{\partial}{\partial x_l}\left[\widehat{J}^{-1}\widehat{F}_{km}\widehat{D}_m\right]\widehat{i}_{kl} - [\![J^{-1}F_{km}D_m]\!]\widehat{n}_k = \widehat{J}^{-1}\widehat{Q}, & \mathbf{x}\in\Gamma \\ \dfrac{\partial}{\partial x_k}\left[J^{-1}F_{km}D_m\right] = 0, & \mathbf{x}\in\mathbb{R}^3\setminus\Omega \\ [\![J^{-1}F_{km}D_m]\!]n_k = 0, & \mathbf{x}\in\partial\Omega \end{cases}, \tag{46}$$

where, again, by $\mathbf{F}$ in $\mathbb{R}^3\setminus\Omega$ we mean any suitably well-behaved extension to $\mathbb{R}^3$ of the deformation gradient $\mathbf{F}$ in the body. Given that $\operatorname{div}\left[J^{-1}\mathbf{F}^T\right] = \widehat{\operatorname{div}}\left[\widehat{J}^{-1}\widehat{\mathbf{F}}^T\right] = \mathbf{0}$, equations (46) simplify to

$$\begin{cases} J^{-1}F_{km}\dfrac{\partial D_m}{\partial x_k} = J^{-1}Q, & \mathbf{x}\in\Omega\setminus\Gamma \\ \widehat{J}^{-1}\widehat{F}_{km}\dfrac{\partial \widehat{D}_m}{\partial x_l}\widehat{i}_{kl} - [\![J^{-1}F_{km}D_m]\!]\widehat{n}_k = \widehat{J}^{-1}\widehat{Q}, & \mathbf{x}\in\Gamma \\ J^{-1}F_{km}\dfrac{\partial D_m}{\partial x_k} = 0, & \mathbf{x}\in\mathbb{R}^3\setminus\Omega \\ [\![J^{-1}F_{km}D_m]\!]n_k = 0, & \mathbf{x}\in\partial\Omega \end{cases}.$$

By employing now the chain rule and the identities $\mathbf{n} = |J\mathbf{F}^{-T}\mathbf{N}|^{-1}\,J\mathbf{F}^{-T}\mathbf{N}$, $\widehat{\mathbf{F}}^{-1} = \mathbf{F}^{-1}\widehat{\mathbf{i}}$, and $\widehat{\mathbf{n}} = \widehat{J}^{-1}\,J\mathbf{F}^{-T}\widehat{\mathbf{N}}$ together with the fact that $J^{\mathtt{i}}\mathbf{F}^{\mathtt{i}-T}\widehat{\mathbf{N}} =$

$J^{\mathtt{m}}\mathbf{F}^{\mathtt{m}-T}\widehat{\mathbf{N}}$, we obtain

$$\begin{cases} \dfrac{\partial D_m}{\partial X_n} F^{-1}_{nk} F_{km} = Q, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \dfrac{\partial \widehat{D}_m}{\partial X_n} \widehat{F}^{-1}_{nk} \widehat{F}_{km} - [\![D_k]\!] \widehat{N}_k = \widehat{Q}, & \mathbf{X} \in \Gamma_0 \\ \dfrac{\partial D_m}{\partial X_n} F^{-1}_{nk} F_{km} = 0, & \mathbf{X} \in \mathbb{R}^3 \setminus \Omega_0 \\ [\![D_k]\!] \widehat{N}_k = 0, & \mathbf{X} \in \partial\Omega_0 \end{cases} .$$

Finally, recognizing that $\widehat{\mathbf{F}}^{-1}\widehat{\mathbf{F}} = \widehat{\mathbf{I}}$, Gauss's law in Lagrangian form (11) readily follows:

$$\begin{cases} \dfrac{\partial D_m}{\partial X_m} = Q, & \mathbf{X} \in \Omega_0 \setminus \Gamma_0 \\ \dfrac{\partial \widehat{D}_m}{\partial X_n} \widehat{I}_{mn} - [\![D_k]\!] \widehat{N}_k = \widehat{Q}, & \mathbf{X} \in \Gamma_0 \\ \dfrac{\partial D_m}{\partial X_m} = 0, & \mathbf{X} \in \mathbb{R}^3 \setminus \Omega_0 \\ [\![D_k]\!] \widehat{N}_k = 0, & \mathbf{X} \in \partial\Omega_0 \end{cases} .$$

Appendix B. Faraday's law in Lagrangian form

Direct use of the definition $\mathbf{E} = \mathbf{F}^T\mathbf{e}$ and the transformation rule $\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{X}$ for material line elements allows to recast the integral form (12) of Faraday's law as

$$\int_{\partial\Sigma} \mathbf{e} \cdot \mathrm{d}\mathbf{x} = \int_{\partial\Sigma_0} \mathbf{E} \cdot \mathrm{d}\mathbf{X} = 0. \tag{47}$$

By making use of Stokes's theorem

$$\int_{\Sigma_0} (\mathrm{Curl}\,\mathbf{E}) \cdot \widetilde{\mathbf{N}}\,\mathrm{d}\mathbf{X} = \int_{\partial\Sigma_0} \mathbf{E} \cdot \mathrm{d}\mathbf{X} + \int_{\partial\mathcal{S}_0} [\![\mathbf{E}]\!] \cdot \mathrm{d}\mathbf{X},$$

this time around in the initial configuration, the Lagrangian localized form (14) of Faraday's law readily follows from (47).

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