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RESEARCH NOTE

An Exact Result for the Macroscopic Response of Porous Neo-Hookean Solids

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Abstract Making use of the particulate microgeometries of Idiart (J. Mech. Phys. Solids 56:2599–2617, 2008), we derive an *exact* and *closed-form* result for the macroscopic response of porous Neo-Hookean solids with *random* microstructures. The stored-energy function is a solution to a Hamilton-Jacobi equation with initial porosity and macroscopic deformation gradient playing the roles of time and space. The main theoretical and practical aspects of the result are discussed.

Keywords Homogenization methods · Porous materials · Finite strain · Polyconvexity

Mathematics Subject Classification (2000) 74B20 · 74M25 · 74Q15 · 74Q20 · 35M99

1 Introduction

This work is concerned with hyperelastic solids containing a *random* and *isotropic* distribution of pores whose characteristic length is much smaller than the specimen size and the scale of variation of the applied loading. In the undeformed configuration, the volume fraction of pores is given by f_0, and the matrix material occupies a domain Ω_0. Under the further

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hypothesis of statistical uniformity, the macroscopic response of the porous solid is formally characterized [1] by the effective stored-energy function

$$\overline{W}(\overline{\mathbf{F}}, f_0) = (1 - f_0) \min_{\mathbf{F}\in\mathcal{K}(\overline{\mathbf{F}})} \frac{1}{|\Omega_0|} \int_{\Omega_0} W(\mathbf{F})\,\mathrm{d}\Omega_0, \tag{1}$$

where W is the stored-energy function characterizing the behavior of the matrix material, and $\mathcal{K}$ denotes the set of kinematically admissible deformation gradients $\mathbf{F}$ with prescribed volume average $\overline{\mathbf{F}}$. Because of the assumed microstructural *randomness* and the physically required *non-convexity* of W in $\mathbf{F}$, computing the solution (assuming that it exists) of the minimum variational problem (1) is a formidable task. Accordingly, very few *exact* results for $\overline{W}$ have been generated thus far [2–5]. *Approximate* results have also remained elusive until very recently (see, e.g., [6–8] and references therein).

In this work, we consider a novel class of particulate microstructures[1] for which the effective stored-energy function (1) can be computed *exactly*: the infinite-rank laminate microstructures of Idiart [9]. Although presented in the context of viscoplasticity, the derivation given in [9] carries over *mutatis mutandis* to hyperelasticity.[2] It follows that the macroscopic stored-energy function $\overline{W}$ of an n-dimensional porous hyperelastic solid with isotropic infinite-rank laminate microstructure is a solution to the Hamilton-Jacobi equation (see Sect. 4.2 in that reference)

$$f_0 \frac{\partial \overline{W}}{\partial f_0} - H\left(\overline{\mathbf{F}}, \overline{W}, \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}\right) = 0, \qquad \overline{W}(\overline{\mathbf{F}}, 1) = 0, \tag{2}$$

where the Hamiltonian is given by

$$H\left(\overline{\mathbf{F}}, \overline{W}, \frac{\partial \overline{W}}{\partial \overline{\mathbf{F}}}\right) = \overline{W} + \frac{1}{|S_n|} \int_{|\mathbf{N}|=1} \max_{\boldsymbol{\omega}} \left[\omega_i \frac{\partial \overline{W}}{\partial \overline{F}_{ij}} N_j - W\left(\overline{\mathbf{F}} + \boldsymbol{\omega} \otimes \mathbf{N}\right) \right] \mathrm{d}S_n(\mathbf{N}) \tag{3}$$

$(i, j = 1, \ldots, n)$. In this last expression, the integral is over the surface of the n-dimensional unit ball S_n, and indicial notation has been employed for clarity.

This note is our first attempt in deriving exact stored-energy functions for porous hyperelastic materials with random microstructures via the Hamilton-Jacobi formulation (2). In this regard, we will restrict the analysis to the case of two-dimensional (2D) porous Neo-Hookean solids. This problem is general enough to contain all the essential features of porous hyperelastic materials, and, at the same time, it is simple enough to permit full analytical treatment.

2 Formulae for Two-Dimensional Porous Neo-Hookean Solids

In the sequel, we examine the case of 2D ($n = 2$ in expression (3)) isotropic porous hyperelastic solids with Neo-Hookean matrix phase of the form

$$W(\mathbf{F}) = \Phi(\lambda_1, \lambda_2) = \begin{cases} \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 - 2) & \text{if } \lambda_1\lambda_2 = 1, \\ +\infty & \text{otherwise,} \end{cases} \tag{4}$$

[1] That is, microstructures in which a discontinuous inclusion phase is embedded in a continuous matrix phase.

[2] This is the case essentially because, much like the effective behavior of viscoplastic laminates, the exact effective behavior of (rank-one) hyperelastic laminates—upon which the derivation is primarily based—corresponds to underlying piecewise-constant strain and stress fields (see, e.g., [3] and [10]).

where the positive material parameter μ stands for the shear modulus in the ground state, and λ_i $(i = 1, 2)$ denote the principal stretches associated with $\mathbf{F}$.

Because of the assumed overall isotropy, it suffices to consider macroscopic diagonal deformations of the form $\overline{\mathbf{F}} = \mathrm{diag}(\overline{\lambda}_1, \overline{\lambda}_2)$. Moreover, the effective stored-energy function of the porous solid can be conveniently expressed as a symmetric function of the macroscopic principal stretches:

$$\overline{W}(\overline{\mathbf{F}}, f_0) = \overline{\Phi}(\overline{\lambda}_1, \overline{\lambda}_2, f_0) = \overline{\Phi}(\overline{\lambda}_2, \overline{\lambda}_1, f_0). \tag{5}$$

Given (4) and (5), the Hamiltonian (3) specializes to

$$H = \overline{\Phi} + \frac{1}{2\pi}\int_0^{2\pi}\Bigg[\omega_1 N_1 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_1} + \omega_2 N_2 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_2} - \frac{\mu}{2}\left((\overline{\lambda}_1 + \omega_1 N_1)^2 + (\overline{\lambda}_2 + \omega_2 N_2)^2 + \omega_2^2 N_1^2 + \omega_1^2 N_2^2 - 2\right)\Bigg]\mathrm{d}\theta, \tag{6}$$

where $N_1 = \cos\theta$ and $N_2 = \sin\theta$, and

$$\omega_1 = -\overline{\lambda}_1 N_1 + \frac{\overline{\lambda}_2 N_1}{\overline{\lambda}_2^{-2} N_1^2 + \overline{\lambda}_1^{-2} N_2^2} + \frac{\overline{\lambda}_1 N_1 N_2^2 (\overline{\lambda}_1 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_1} - \overline{\lambda}_2 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_2})}{\mu(\overline{\lambda}_2^{-2} N_1^2 + \overline{\lambda}_1^{-2} N_2^2)},$$

$$\omega_2 = -\overline{\lambda}_2 N_2 + \frac{\overline{\lambda}_1 N_2}{\overline{\lambda}_2^{-2} N_1^2 + \overline{\lambda}_1^{-2} N_2^2} - \frac{\overline{\lambda}_2 N_2 N_1^2 (\overline{\lambda}_1 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_1} - \overline{\lambda}_2 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_2})}{\mu(\overline{\lambda}_2^{-2} N_1^2 + \overline{\lambda}_1^{-2} N_2^2)} \tag{7}$$

are the components of the maximizing vector $\boldsymbol{\omega}$. Upon explicit evaluation of the integral in (6), together with some algebraic manipulation, the Hamilton-Jacobi equation (2) finally reduces to

$$f_0 \frac{\partial\overline{\Phi}}{\partial f_0} - \overline{\Phi} + \frac{\overline{\lambda}_1^{-2} + \overline{\lambda}_1\overline{\lambda}_2 - 2}{2(\overline{\lambda}_1 + \overline{\lambda}_2)}\frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_1} + \frac{\overline{\lambda}_2^{-2} + \overline{\lambda}_1\overline{\lambda}_2 - 2}{2(\overline{\lambda}_1 + \overline{\lambda}_2)}\frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_2} - \frac{(\overline{\lambda}_1 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_1} - \overline{\lambda}_2 \frac{\partial\overline{\Phi}}{\partial\overline{\lambda}_2})^2}{4(\overline{\lambda}_1 + \overline{\lambda}_2)^2\mu} = -\frac{2 + \overline{\lambda}_1\overline{\lambda}_2(\overline{\lambda}_1^{-2} + \overline{\lambda}_2^{-2} - 4)}{4\overline{\lambda}_1\overline{\lambda}_2}\mu, \tag{8}$$

with initial condition $\overline{\Phi}(\overline{\lambda}_1, \overline{\lambda}_2, 1) = 0$. This *nonlinear* first-order partial differential equation admits the following analytical solution:

$$\overline{\Phi} = \frac{\mu}{2}\frac{1 - f_0}{1 + f_0}\left[\overline{\lambda}_1^{-2} + \overline{\lambda}_2^{-2} - 2\overline{\lambda}_1\overline{\lambda}_2\right] + \frac{\mu}{2}(\overline{\lambda}_1\overline{\lambda}_2 - 1)\ln\left[\frac{\overline{\lambda}_1\overline{\lambda}_2 + f_0 - 1}{f_0\overline{\lambda}_1\overline{\lambda}_2}\right]. \tag{9}$$

Expression (9) *constitutes an* exact *and* closed-form *stored-energy function for 2D isotropic porous Neo-Hookean solids, with infinite-rank laminate microstructures, subjected to general loading conditions*.

Following are a few theoretical and practical remarks regarding the above result:

(i) In the limit of small deformations ($\overline{\mathbf{F}} \to \mathbf{I}$), the stored-energy function (9) takes the form

$$\overline{\Phi} = \overline{\mu}\left(\overline{\varepsilon}_1^2 + \overline{\varepsilon}_2^2\right) + \frac{\overline{\kappa} - \overline{\mu}}{2}(\overline{\varepsilon}_1 + \overline{\varepsilon}_2)^2 + O(\|\overline{\mathbf{F}} - \mathbf{I}\|^3), \tag{10}$$

where $\overline{\varepsilon}_i \equiv \overline{\lambda}_i - 1$ $(i = 1, 2)$, and

$$\overline{\kappa} = \mu \frac{1 - f_0}{f_0}, \qquad \overline{\mu} = \mu \frac{1 - f_0}{1 + f_0}, \tag{11}$$

are, respectively, the effective bulk and shear moduli in the ground state. Expressions (11) agree *exactly* with the Hashin-Shtrikman (HS) bounds for the bulk and shear moduli of 2D isotropic porous materials with linearly elastic incompressible matrix, *cf.* expressions (4.26) and (4.28) in [11]. These bounds are known to provide accurate approximations to the overall moduli of porous materials with particulate microstructures, at least for small to moderate porosities [12]. And in the limit of vanishingly small porosities ($f_0 \to 0$), they recover the exact overall response of a solid containing a dilute distribution of circular pores.

(ii) Interestingly, the connection with the HS bounds is not restricted to small deformations. Indeed, under finite isochoric deformations ($\overline{\lambda}_1 = \overline{\lambda}, \overline{\lambda}_2 = \overline{\lambda}^{-1}$), expression (9) reduces to

$$\overline{\Phi}(\overline{\lambda}, \overline{\lambda}^{-1}, f_0) = \frac{\mu}{2} \frac{1 - f_0}{1 + f_0} \left[\overline{\lambda}^2 + \overline{\lambda}^{-2} - 2\right], \tag{12}$$

which is seen to have the same functional form as the Neo-Hookean matrix phase, with a shear modulus given by the linear HS bound (11).[3]

(iii) Under hydrostatic deformations ($\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}$), the stored-energy function (9) simplifies to

$$\overline{\Phi}(\overline{\lambda}, \overline{\lambda}, f_0) = \frac{\mu}{2}(\overline{\lambda}^2 - 1) \ln \left[\frac{\overline{\lambda}^2 + f_0 - 1}{f_0 \overline{\lambda}^2}\right]. \tag{13}$$

This expression agrees *exactly* with the stored-energy function of porous Neo-Hookean solids with 'hollow cylinder assemblage' (HCA) microgeometries [2]—*cf.* expression (69) in [6]. HCAs are particulate microgeometries commonly used to model porous materials, presumably because their stored-energy function can be computed exactly for hydrostatic deformations. Expression (9) can thus be thought of as an extension of such an exact result for HCAs to non-isotropic deformations.

(iv) The stored-energy function (9) can be rewritten as

$$\overline{\Phi} = g(\overline{\mathbf{F}}) + h(\det \overline{\mathbf{F}}), \tag{14}$$

where

$$g(\overline{\mathbf{F}}) = \frac{\mu}{2} \frac{1 - f_0}{1 + f_0} (\overline{\mathbf{F}} \cdot \overline{\mathbf{F}} - 2), \tag{15}$$

and

$$h(\det \overline{\mathbf{F}}) = \frac{\mu}{2}(\det \overline{\mathbf{F}} - 1) \left[\ln\left(\frac{\det \overline{\mathbf{F}} + f_0 - 1}{f_0 \det \overline{\mathbf{F}}}\right) - 2\frac{1 - f_0}{1 + f_0}\right] \tag{16}$$

are strictly convex functions of their arguments. Thus, the stored-energy function (9) is strictly polyconvex, and therefore *strictly rank-one convex* (or strongly elliptic). A direct physical implication of this type of convexity is that the result (9) does *not* account

[3] deBotton [3] obtained this same result assuming that the porous phase was incompressible.

for possible softening mechanisms leading to the development of long-wavelength instabilities [13].

(v) While exact for a specific microstructure, we conjecture that (9) is actually a rigorous upper bound for the effective stored-energy functions of general 2D isotropic porous Neo-Hookean materials. In line with [9, 14], this conjecture is primarily based on the observation that, for any applied macroscopic loading $\overline{\mathbf{F}}$, the underlying porous phase in the infinite-rank laminate undergoes a *uniform* deformation—such class of deformations is usually associated with the stiffest possible response of soft materials. Note that remarks (i) and (iv) above directly support that the stored-energy function (9) indeed constitutes an upper bound. The validity of this conjecture would make expression (9) the first non-trivial bound for porous hyperelastic solids subjected to general finite deformations, paralleling the linear bounds of Hashin and Shtrikman.

3 Sample Results and Discussion

With the objective of providing further insight into the new result (9), henceforth referred to as HAM, we confront it in this section with the 'linear-comparison' (LC) estimates[4] of Lopez-Pamies and Ponte Castañeda [6], the finite element (FE) simulations of Moraleda et al. [15], and the Voigt bound of Ogden [16]. Here, it is recalled that the LC estimates are appropriate for a polydisperse, isotropic distribution of initially circular pores, while the FE results correspond to a monodisperse and approximately isotropic distribution of initially circular pores. On the other hand, the rigorous Voigt upper bound—non-trivial only for macroscopic isochoric deformations—is independent of the actual distribution of pores.

Figures 1(a) and (b) show plots for the effective stored-energy function $\overline{\Phi}$ and associated stresses $\overline{t}_i = \partial\overline{\Phi}/\partial\overline{\lambda}_i$ $(i = 1, 2)$ for isochoric deformations $(\overline{\lambda}_1 = \overline{\lambda}_2^{-1} = \overline{\lambda})$, as a function of the applied stretch $\overline{\lambda}$. The results correspond to initial porosities of $f_0 = 0.05$ and 0.2. It is observed from Fig. 1(a) that the HAM stored-energy function satisfies, of course, the rigorous Voigt upper bound. Moreover, the HAM and LC results are seen to be essentially identical. The agreement between the HAM and LC stresses shown in Fig. 1(b) is also remarkably good, especially for the tensile stress $\overline{t}_1$.

Figures 1(c) and (d) show plots for the macroscopic stress $\overline{t} = \overline{t}_1 = \overline{t}_2$ for hydrostatic compression $(\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda} \leq 1)$ and hydrostatic tension $(\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda} \geq 1)$, respectively, as a function of the applied stretch $\overline{\lambda}$. The results correspond to initial porosities of $f_0 = 0.05, 0.1, 0.2$. In the case of tension, all three results (HAM, LC, FE) lead to very similar responses for small to moderate deformations, especially for larger values of initial porosity. For large deformations, on the other hand, the HAM response is seen to be in between the LC and FE results. The stiffer response of the FE simulations relative to the HAM result is inconsistent with the conjectured upper bound character of (9). A possible explanation of this result is quite simply that (9) is not an upper bound as we have conjectured. However, the overly stiff behavior of the FE results may also be attributed to the fact that the simulations correspond to microstructures that are isotropic only approximately. Further analysis is required to clarify this point. In the case of compression, the agreement between the HAM and LC stress-stretch results is seen to be very good, with the LC estimates exhibiting a slightly softer response. This softer behavior is consistent with the fact that, unlike the stored-energy function (9), the LC estimates lose strong ellipticity (as denoted with the symbol " $\circ$ " in the plots) under hydrostatic compressive loadings.

[4] As given by equation (60) in that reference.

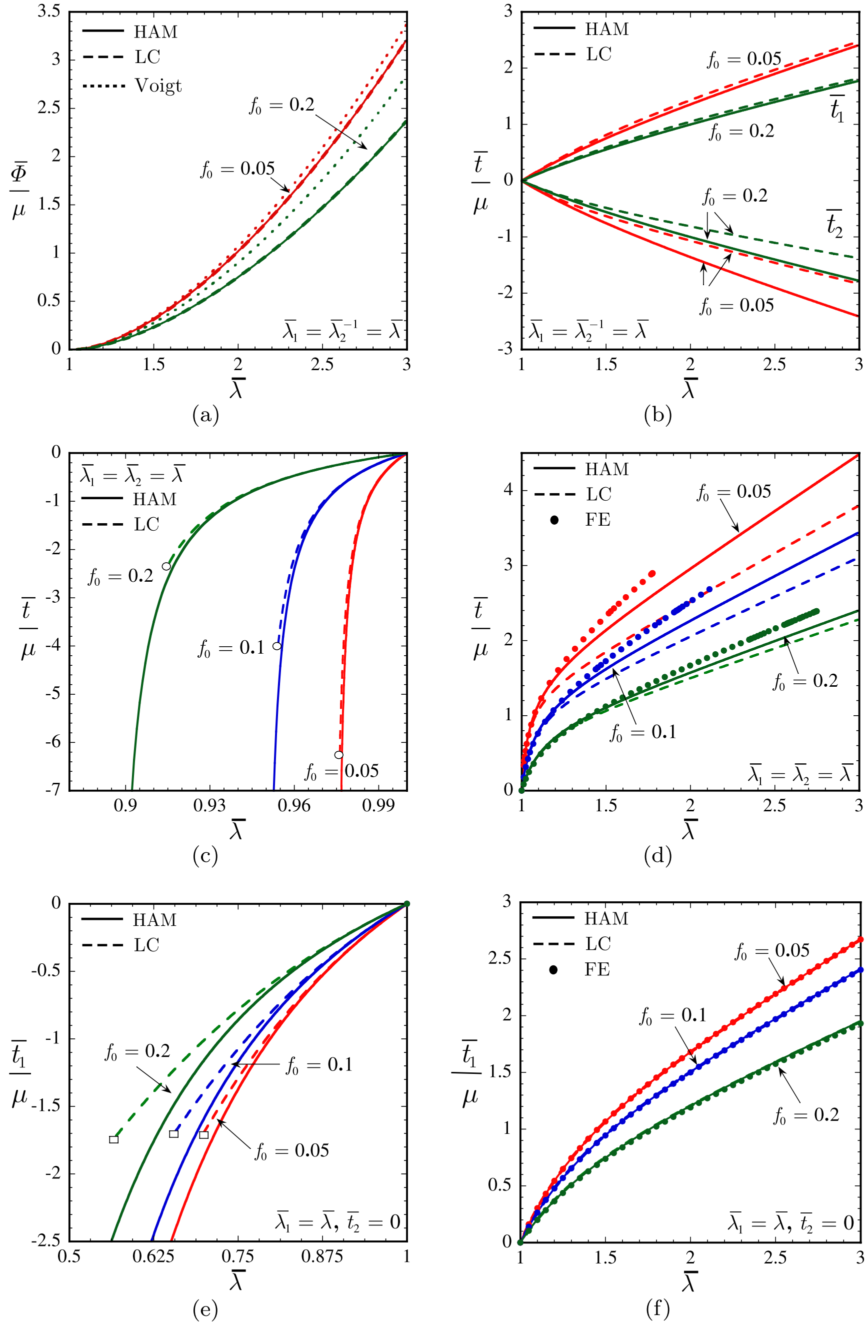

Fig. 1 Macroscopic response of isotropic porous Neo-Hookean solids with random microstructures and various values of initial porosity f_0. (**a**, **b**) isochoric deformation $\overline{\lambda}_1 = \overline{\lambda}$, $\overline{\lambda}_2 = \overline{\lambda}^{-1}$, (**c**, **d**) hydrostatic deformation $\overline{\lambda}_1 = \overline{\lambda}_2 = \overline{\lambda}$, and (**e**, **f**) uniaxial traction $\overline{\lambda}_1 = \overline{\lambda}$ with $\overline{t}_2 = \partial\overline{\Phi}/\partial\overline{\lambda}_2 = 0$. Results are shown for the effective stored-energy function, $\overline{\Phi}$, and associated stresses, $\overline{t}_i = \partial\overline{\Phi}/\partial\overline{\lambda}_i$ $(i = 1, 2)$, for the HAM constitutive relation (9) (*solid line*), the LC estimates of Lopez-Pamies and Ponte Castañeda [6] (*dashed line*), the FE simulations of Moraleda et al. [15] (*dots*), and the Voigt upper bound (*dotted line*)

Finally, Figs. 1(e) and (f) display plots for the stress $\bar{t}_1$ for uniaxial compression ($\bar{\lambda}_1 = \bar{\lambda} \leq 1, \bar{t}_2 = 0$) and uniaxial tension ($\bar{\lambda}_1 = \bar{\lambda} \geq 1, \bar{t}_2 = 0$), respectively, as a function of $\bar{\lambda}$. In the case of tension, all three results (HAM, LC, FE) are in excellent agreement for all values of initial porosity considered. In the case of compression, the HAM and the LC results are in fair agreement, with the LC estimates exhibiting a progressively softer behavior with increasing deformation. An interesting feature that further differentiates the result (9) from the LC estimates is that the LC estimates predict the closure of porosity at finite values of the applied compressive stretch (as denoted with the symbol " □ " in the plots), while the constitutive relation (9) predicts that the porosity never reaches zero value under uniaxial compressive loadings with $\bar{t}_2 = 0$.

In summary, the comparisons shown in Fig. 1 confirm that the exact stored-energy function (9) does characterize the macroscopic response of 2D isotropic porous Neo-Hookean solids with *random* and *particulate* microstructures. These encouraging results motivate further efforts to exploit the Hamilton-Jacobi formulation (2) in order to generate exact results for broader material systems.

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