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On the Extremal Properties of Hashin's Hollow Cylinder Assemblage in Nonlinear Elasticity

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Abstract Counterexamples are given of 2D isotropic porous nonlinear elastic solids whose responses under hydrostatic loading are stiffer than that of the corresponding hollow cylinder assemblage. In addition to disproving existing conjectures in the literature, the results serve to illustrate that radially symmetric deformation—contrary to common belief—is not necessarily the stiffest possible deformation mechanism to accommodate hydrostatic loading. Instead, depending on the growth conditions of the material at hand, other types of deformations may lead to stiffer responses.

Keywords Porous materials · Finite strain · Microstructures

Mathematics Subject Classification (2000) 74B20 · 74M25 · 74Q15 · 74Q20 · 74S05

1 Introduction

It is well known that the 'hollow cylinder assemblage' (HCA) of Hashin [1] is an extremal microstructure that attains the Hashin-Shtrikman upper bound [2] for the effective (in-plane) bulk modulus of porous materials with *linear elastic* isotropic matrix phase containing an isotropic, but otherwise arbitrary, distribution of porosity.[1] In the literature [3, 4], it has been

[1] The same is true for the corresponding 3D case of a 'hollow sphere assemblage'.

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conjectured that the radially symmetric response of a HCA remains as well the stiffest equilibrium response of isotropic porous *nonlinear elastic* materials under hydrostatic loading conditions; see also [5–7] for similar assertions in the context of *viscoplasticity*. In this Note, with the help of recent analytical [4] and numerical [8] techniques, we disprove this conjecture by providing counterexamples of isotropic porous materials whose responses under hydrostatic loading are stiffer than that of the corresponding HCA.

The subsequent analysis is set in the theoretical framework of nonlinear elastic composites at finite strain [9]. We focus on 2D isotropic nonlinear elastic solids containing a *random* and *isotropic* distribution of vacuous pores whose characteristic length is much smaller than the specimen size and the scale of variation of the applied loading. In the undeformed configuration, the initial volume fraction of pores (or initial porosity) is given by f_0 and the matrix occupies a spatial domain Ω_m. The constitutive behavior of the matrix is characterized by an isotropic stored-energy function $W(\mathbf{F}) = \phi(\lambda_1, \lambda_2)$ with λ_1 and λ_2 denoting the singular values of the deformation gradient tensor $\mathbf{F}$. For consistency with the theory of linear elasticity, we require that $\phi(\lambda_1, \lambda_2) = \mu((\lambda_1 - 1)^2 + (\lambda_2 - 1)^2) + (\kappa - \mu)(\lambda_1 + \lambda_2 - 2)^2/2 + O(\|\mathbf{F} - \mathbf{I}\|^3)$ as $\mathbf{F} \to \mathbf{I}$, where the positive material parameters μ and κ stand for, respectively, the shear and (in-plane) bulk moduli of the matrix in its undeformed state. Following Hill [9], the macroscopic or overall response of the above-described porous materials under hydrostatic loading conditions is formally characterized by the effective stored-energy function

$$\overline{\phi}(\overline{\lambda}, f_0) = (1 - f_0) \min_{\mathbf{F} \in \mathcal{K}(\overline{\mathbf{F}})} \frac{1}{|\Omega_m|} \int_{\Omega_m} \phi(\lambda_1, \lambda_2)\, \mathrm{d}\Omega_m, \tag{1}$$

where $\mathcal{K}$ denotes a suitable set of admissible deformation gradient tensors $\mathbf{F}(\mathbf{X})$ with prescribed volume average $\overline{\mathbf{F}} = \overline{\lambda}\mathbf{I}$. Because of the possible occurrence of instabilities, the solution to the Euler-Lagrange equations associated with (1) need *not* be unique. In deriving tractable overall responses of porous nonlinear elastic materials, it is thus common practice (see, e.g., [10–12]) to carry out the minimization in (1) over restricted sets $\mathcal{K}$ of admissible deformations that exclude bifurcation solutions associated with local instabilities (see [13] for a detailed discussion of this issue). The specific choices of the sets $\mathcal{K}$ that we utilize in this work are described in the sequel.

2 Hydrostatic Response of the HCA

When the microstructure is taken to be a HCA, as depicted in Fig. 1, and the admissible deformations are restricted to be radially symmetric in each hollow cylinder, the solution to (1) can be shown to coincide identically with that of a corresponding single cylindrical shell [14]. In this case, the effective stored-energy function $\overline{\phi}$ takes the form

$$\overline{\phi}_{HCA}(\overline{\lambda}, f_0) = 2 \int_{f_0^{1/2}}^{1} \phi\left(r', \frac{r}{R}\right) R\,\mathrm{d}R, \tag{2}$$

where $r = r(R)$ is the solution to the nonlinear ordinary differential equation (ode)

$$\frac{\mathrm{d}}{\mathrm{d}R}\left[R\phi_1\left(r', \frac{r}{R}\right)\right] - \phi_2\left(r', \frac{r}{R}\right) = 0, \tag{3}$$

subject to the boundary conditions

$$r(1) = \overline{\lambda} \quad \text{and} \quad \phi_1\left(r'(f_0^{1/2}), \frac{r(f_0^{1/2})}{f_0^{1/2}}\right) = 0, \tag{4}$$

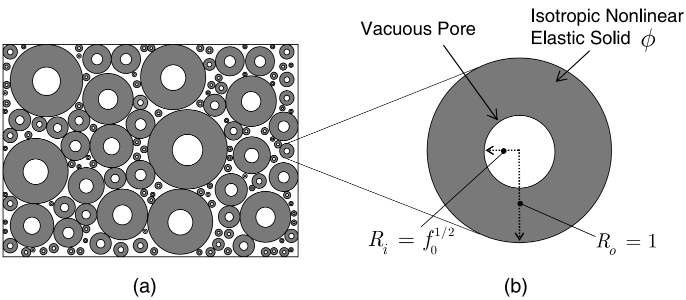

Fig. 1 Undeformed-configuration schematics of (**a**) Hashin's hollow cylinder assemblage and (**b**) the corresponding representative cylindrical shell with outer radius $R_o = 1$ and inner radius $R_i = f_0^{1/2}$, where the quantity f_0 denotes the initial porosity in the material (i.e., $f_0 = R_i^2/R_o^2$)

that minimizes the right-hand side of (2); the notation $r' = \mathrm{d}r/\mathrm{d}R$ and $\phi_i = \partial\phi/\partial\lambda_i$ $(i = 1, 2)$ has been used for convenience. Here it is relevant to remark that, by definition, the effective stored-energy function (2) *bounds from above* the effective stored-energy function (1) that would result by minimizing over the set of all (possibly non-radially symmetric) deformations $\mathbf{F}(\mathbf{X})$. The radially symmetric deformation (3)–(4) in each hollow cylinder is, however, the global minimizer of (1) within a sufficiently small neighborhood of applied stretches $\overline{\lambda}$ containing the point $\overline{\lambda} = 1$. Whether such a deformation field remains the global energy minimizer for arbitrarily large $\overline{\lambda}$ is still an open problem (expected to strongly depend on the convexity properties and growth conditions[1] of ϕ and on the value of f_0).

Now, the nonlinear ode (3) does not generally admit explicit solutions. However, (3) can be solved in closed form in the limit of small deformations as $\overline{\lambda} \to 1$, and the integral in (2) carried out explicitly to render

$$\overline{\phi}_{HCA}\left(\overline{\lambda}, f_0\right) = 2\overline{\kappa}_{HCA}\left(\overline{\lambda} - 1\right)^2 + O\left((\overline{\lambda} - 1)^3\right) \quad \text{with } \overline{\kappa}_{HCA} = \frac{(1 - f_0)\kappa}{f_0\kappa + \mu}\mu. \tag{5}$$

As anticipated above, expression $(5)_2$ for the effective bulk modulus $\overline{\kappa}_{HCA}$ of the HCA in its undeformed state is seen to agree exactly with the Hashin-Shtrikman upper bound for the bulk modulus of isotropic porous linear elastic materials (*cf.* expression (4.26) in [15]). Another limiting case of practical interest for which (2) simplifies to an explicit form is that of *incompressible solids* when $\phi(\lambda_1, \lambda_2) = +\infty$ if $\lambda_1\lambda_2 \neq 1$. In this case, the constraint of incompressibility implies that $r(R) = (R^2 + \overline{\lambda}^2 - 1)^{1/2}$ irrespectively of the specific choice of stored-energy function ϕ. This allows to rewrite the effective stored-energy function (2), after a little manipulation, as follows

$$\overline{\phi}_{HCA}(\overline{\lambda}, f_0) = 2(\overline{\lambda}^2 - 1)\int_{\overline{\lambda}}^{(1+\frac{\overline{\lambda}^2-1}{f_0})^{1/2}} \frac{z}{(z^2-1)^2}\phi\left(z^{-1}, z\right)\mathrm{d}z. \tag{6}$$

[1] For a broad class of isotropic polyconvex materials, Sivaloganathan and Spector [16, 17] have recently shown that radially symmetric deformations are actual energy minimizers for the elastostatics problem of a single shell under hydrostatic loading. These results might possibly contain some implications for an assemblage of shells.

In the limit of small deformations as $\overline{\lambda} \to 1$, expression (6) simplifies to

$$\overline{\phi}_{HCA}\left(\overline{\lambda}, f_0\right) = 2\overline{\kappa}^{I}_{HCA}\left(\overline{\lambda}-1\right)^2 + O\left((\overline{\lambda}-1)^3\right) \quad \text{with } \overline{\kappa}^{I}_{HCA} = \frac{1-f_0}{f_0}\mu, \tag{7}$$

which corresponds to $\kappa = +\infty$ in (5). For later reference, we finally spell out the further specialization of (6) to incompressible Neo-Hookean solids with stored-energy function

$$\phi(\lambda_1, \lambda_2) = \begin{cases} \frac{\mu}{2}(\lambda_1^2 + \lambda_2^2 - 2) & \text{if } \lambda_1\lambda_2 = 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{8}$$

The result can be written in closed form and reads simply as

$$\begin{aligned} \overline{\phi}^{neo}_{HCA}\left(\overline{\lambda}, f_0\right) &= 2(\overline{\lambda}^2 - 1)\int_{\overline{\lambda}}^{(1+\frac{\overline{\lambda}^2-1}{f_0})^{1/2}} \frac{z}{(z^2-1)^2}\frac{\mu}{2}\left(z^{-2} + z^2 - 2\right)\mathrm{d}z \\ &= \frac{\mu}{2}(\overline{\lambda}^2 - 1)\ln\left[\frac{\overline{\lambda}^2 + f_0 - 1}{\overline{\lambda}^2 f_0}\right]. \end{aligned} \tag{9}$$

In the literature [3, 4], it has been conjectured that the HCA result (2) is an upper bound for the effective stored-energy function $\overline{\phi}$ of isotropic porous nonlinear elastic materials (with any isotropic distribution of pores) based on the facts that: (i) $\overline{\phi}_{HCA}$ is indeed a rigorous upper bound for small deformations and (ii) the deformation gradient field within the underlying pores in the HCA is *uniform* (i.e., the initially circular pores deform *with radial symmetry* into pores that are also of circular shape) and such a class of deformations is usually associated with the stiffest possible response of soft materials.

3 An Isotropic Porous Material with Stiffer Hydrostatic Response than that of the HCA

Consider now an isotropic porous material made up of a HCA where the representative cylindrical shell has inner radius $R_i = v_0^{1/2}$ and outer radius $R_o = 1$ in the undeformed configuration, as illustrated in Figs. 2(a) and 2(b). The shell material consists of an incompressible Neo-Hookean matrix with stored-energy function (8) that contains (see Fig. 2(c)) an

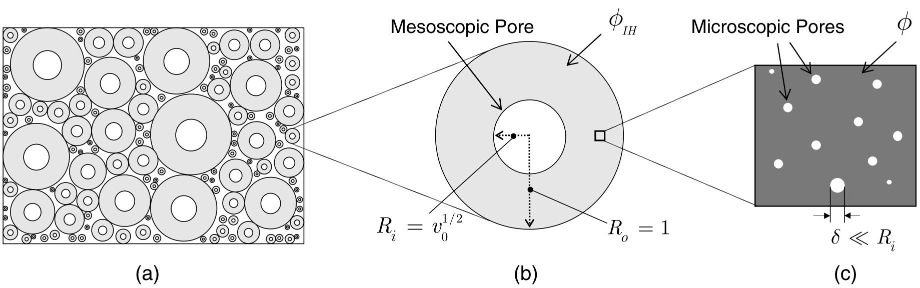

Fig. 2 Undeformed-configuration depiction of an isotropic porous material with two populations of pores at different length scales. At the 'mesoscale' the material is made up of an assemblage of hollow cylinders with initial porosity $v_0 = R_i^2/R_o^2$. At the 'microscale', the material in the cylindrical shells is made up of an incompressible Neo-Hookean matrix that contains an isotropic distribution of disconnected pores with initial porosity c_0. The total initial porosity in the material is given by $f_0 = v_0 + (1 - v_0)c_0$

isotropic distribution of disconnected pores of initial volume fraction c_0; the total initial porosity in the material is hence given by

$$f_0 = v_0 + (1 - v_0)c_0. \tag{10}$$

The 'microscopic' pores within the Neo-Hookean matrix are much smaller in size than the 'mesoscopic' pores in the shells, so that the shell material can be regarded as homogeneous in the HCA. Here, we consider in particular that the shell material is made up of the porous Neo-Hookean solid recently constructed by Lopez-Pamies and Idiart [4]—via an iterated homogenization method [18] together with a novel result for sequential laminates [19]—for which the homogenized stored-energy function can be computed exactly and takes the simple form

$$\phi_{IH}(\lambda_1, \lambda_2) = \frac{\mu}{2}\frac{1-c_0}{1+c_0}\left(\lambda_1^2 + \lambda_2^2 - 2\lambda_1\lambda_2\right) + \frac{\mu}{2}(\lambda_1\lambda_2 - 1)\ln\left[\frac{\lambda_1\lambda_2 + c_0 - 1}{c_0\lambda_1\lambda_2}\right]. \tag{11}$$

Because the underlying 'mesostructure' is nothing more than a HCA, the effective stored-energy function of the above-devised porous material under hydrostatic loading can then be computed by suitably adapting equations (2)–(4). The result can be written as

$$\begin{aligned}\overline{\phi}_{IH}^{neo}\left(\overline{\lambda}, f_0\right) = 2\int_{v_0^{1/2}}^{1}\Bigg\{&\frac{\mu}{2}\frac{1-c_0}{1+c_0}\left(r'^2 + \frac{r^2}{R^2} - 2\frac{r'r}{R}\right)\\ &+\frac{\mu}{2}\left(\frac{r'r}{R} - 1\right)\ln\left[\frac{\frac{r'r}{R} + c_0 - 1}{c_0\frac{r'r}{R}}\right]\Bigg\}R\,\mathrm{d}R,\end{aligned} \tag{12}$$

where $r = r(R)$ is solution to the second-order nonlinear ode

$$\begin{aligned}&rR^2r''\left[R^2\left(c_0^2 + 2(c_0-1)^2r'^2 - 1\right) + rRr'\left(4(c_0-1)r'^2 + (c_0+1)^2\right) + 2r^2r'^4\right]\\ &\quad + r'\left(Rr' - r\right)\left[\left(c_0^2 - 1\right)R^3 + rr'\left(2rr'\left(2(c_0-1)R + rr'\right) + (c_0(3c_0-2)+3)R^2\right)\right] = 0\end{aligned} \tag{13}$$

subject to the boundary conditions

$$r(1) = \overline{\lambda} \tag{14}$$

and

$$\begin{aligned}&\frac{2(c_0-1)(r(\sqrt{v_0}) - \sqrt{v_0}r'(\sqrt{v_0}))}{(c_0+1)\sqrt{v_0}} - \frac{c_0 r(\sqrt{v_0})}{(c_0-1)\sqrt{v_0} + r(\sqrt{v_0})r'(\sqrt{v_0})}\\ &\quad + \frac{r(\sqrt{v_0})\ln\left[\frac{(c_0-1)\sqrt{v_0}}{c_0 r(\sqrt{v_0})r'(\sqrt{v_0})} + \frac{1}{c_0}\right]}{\sqrt{v_0}} + \frac{1}{r'(\sqrt{v_0})} = 0.\end{aligned} \tag{15}$$

Here again it is relevant to remark that, similar to (2), the effective stored-energy function (12) has been constructed by minimizing (1) over a restricted set of admissible deformations, one in which the deformation of each mesoscopic shell is radially symmetric. Accordingly, expression (12) corresponds to the total elastic energy (per unit undeformed volume) associated with a solution $\mathbf{F}(\mathbf{X})$ of equilibrium which is not necessarily the global minimizer of (1).

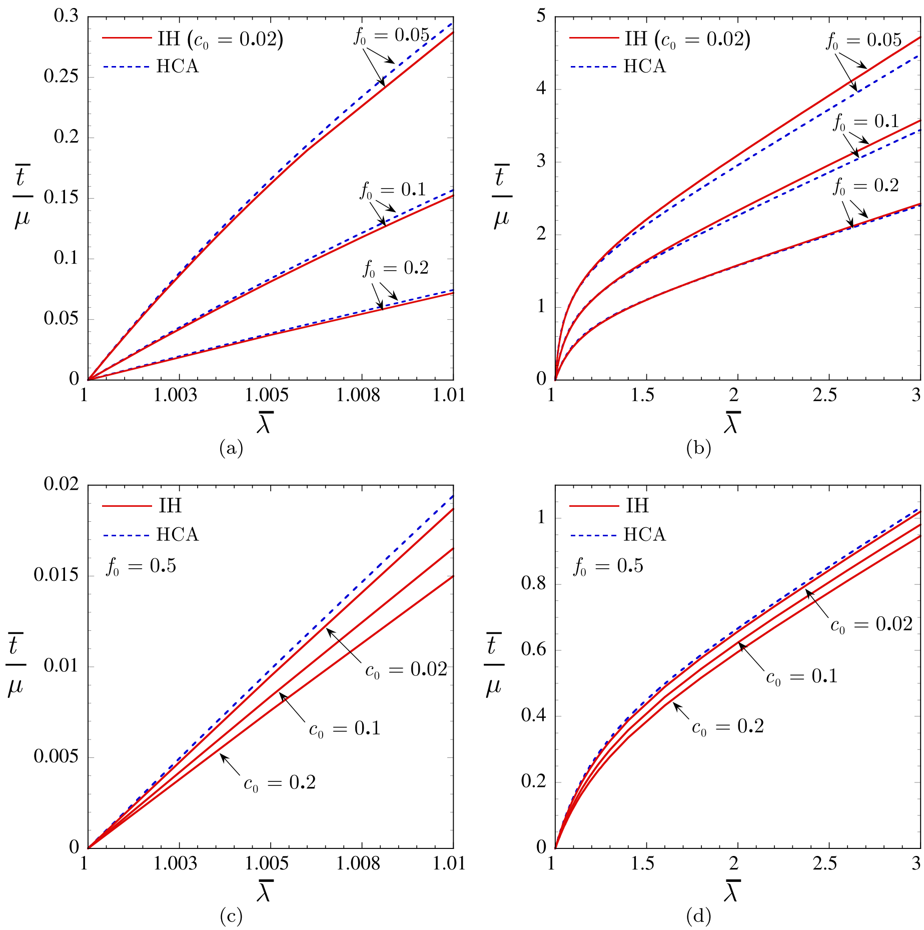

Fig. 3 Overall hydrostatic response of two types of isotropic porous Neo-Hookean materials with the same initial porosity f_0. The *solid lines* correspond to the two-population-of-pores material with effective stored-energy function (12) and the *dashed lines* correspond to the HCA with effective stored-energy function (9). Results are shown in terms of the overall nominal stress $\overline{t}/\mu = 1/(2\mu)\mathrm{d}\overline{\phi}/\mathrm{d}\overline{\lambda}$ as a function of the applied hydrostatic deformation $\overline{\lambda}$. Parts (**a**) and (**c**) show results in the small-deformation regime ($1 \leq \overline{\lambda} \leq 1.01$), while parts (**b**) and (**d**) show the corresponding results for large deformations ($1 \leq \overline{\lambda} \leq 3$)

In the limit of *small deformations* as $\overline{\lambda} \to 1$, it is an easy matter to show that the effective stored-energy function (12) reduces to

$$\overline{\phi}^{neo}_{IH}\left(\overline{\lambda}, f_0\right) = 2\overline{\kappa}_{IH}\left(\overline{\lambda}-1\right)^2 + O\left((\overline{\lambda}-1)^3\right) \quad \text{with } \overline{\kappa}_{IH} = \frac{(1-c_0)(1-f_0)}{f_0 + c_0(f_0 - 2c_0)}\mu, \qquad (16)$$

where relation (10) has been utilized. Since $0 \leq c_0 \leq f_0 \leq 1$, we have that $\overline{\kappa}_{IH} \leq \overline{\kappa}^{I}_{HCA}$ and hence that $\overline{\phi}^{neo}_{IH} \leq \overline{\phi}^{neo}_{HCA}$ in the small-deformation regime, as expected. For *large deformations*, the effective stored-energy function (12) cannot be written in closed form since the ode (13) subject to (14) and (15) does not admit an explicit solution. At any rate, it is straightforward to solve numerically this boundary-value problem, for instance, with the shooting method which is particularly well suited for this class of problems (see, e.g., [20]). Figures 3(a) and 3(b) show sample solutions for values of total and 'microscopic' initial porosities, f_0 and c_0, for which $\overline{\phi}_{IH}^{neo} > \overline{\phi}_{HCA}^{neo}$ in the large-deformation regime; for clarity of presentation, the results are plotted in terms of the derivative[2] $\overline{t}/\mu = 1/(2\mu)\mathrm{d}\overline{\phi}/\mathrm{d}\overline{\lambda}$ as a function of the applied deformation $\overline{\lambda}$ in the small (Fig. 3(a)) and large (Fig. 3(b)) deformation regimes. *These results thus disprove the conjecture that the radially symmetric response (2) of a HCA bounds from above the response of isotropic porous nonlinear elastic materials (with any isotropic distribution of pores).*

In order to gain insight into why the HCA result (2) is *not* an upper bound, we begin by remarking from Fig. 3(b) that the response of the two-population-of-pores material—in which the deformation gradient field in the pores is *not uniform*—is increasingly stiffer than the HCA response for decreasing values of total initial porosity f_0; for sufficiently large f_0, it remains in fact softer than the HCA response for all applied deformations $\overline{\lambda}$, as illustrated in parts (c) and (d) of Fig. 3. Now, because of the incompressibility of the Neo-Hookean stored-energy function (8), smaller values of initial porosity f_0 at fixed $\overline{\lambda}$ imply significantly larger local deformations—that become in fact unbounded in the limit as $f_0 \to 0+$—in the matrix material surrounding the pores. The above results then suggest that the stiffest possible hydrostatic response of an isotropic porous material depends sensitively on the growth conditions[3] of its matrix phase, and not just on whether the deformation is radially symmetric as commonly believed. In this regard, we note that the Neo-Hookean material has relatively strong growth conditions in the sense that it does not admit radially symmetric cavitation in 2D (see Sect. 5 in [21]). The question arises then as to whether the HCA response (2) might still remain an upper bound for the response of porous materials with matrices that have different growth conditions than (8). This issue is explored in the next section.

4 Further Counterexamples Based on FE Simulations

We consider here an isotropic porous material made up of a random and isotropic distribution of initially circular pores with monodisperse sizes embedded in an isotropic matrix phase with stored-energy function [23]

$$\phi(\lambda_1, \lambda_2) = \begin{cases} \frac{3^{1-\alpha}}{2\alpha}\mu\left[\left(\lambda_1^2 + \lambda_2^2 + 1\right)^{\alpha} - 3^{\alpha}\right] & \text{if } \lambda_1\lambda_2 = 1, \\ +\infty & \text{otherwise.} \end{cases} \tag{17}$$

In this last expression, the real-valued parameter α modulates the growth conditions of the matrix: the larger the value of α the stronger the growth conditions; note that for $\alpha = 1$, expression (17) reduces to the Neo-Hookean stored-energy function (8). The overall hydrostatic response of this type of porous material is computed here by means of the finite-element (FE) approach recently implemented by Moraleda et al. [8]. For a detailed description of the analysis we refer to [8]. In the present context, it is appropriate to mention that this approach models the randomness and isotropy of the microstructure by means of an infinite periodic medium where the repeated unit cell contains a large but *finite* number of pores—generated via an appropriate algorithm that ensures a random and homogeneous distribution [22]—and that as a result the microstructure is (not exactly but) only approx-

[2] Here, we note that $\mathrm{d}\overline{\phi}_{IH}^{neo}/\mathrm{d}\overline{\lambda} > \mathrm{d}\overline{\phi}_{HCA}^{neo}/\mathrm{d}\overline{\lambda}$ implies $\overline{\phi}_{IH}^{neo} > \overline{\phi}_{HCA}^{neo}$ for large $\overline{\lambda}$.

[3] That is, the behavior of matrix stored-energy function $W(\mathbf{F})$ as $\|\mathbf{F}\| \to +\infty$.

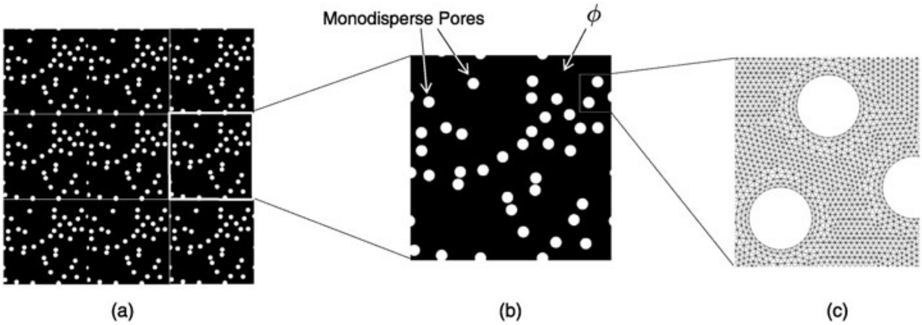

Fig. 4 Finite element model—in the undeformed configuration—for a realization of a porous material made up of a random and isotropic distribution of circular pores with monodisperse sizes embedded in a nonlinear elastic matrix. Part (**a**) depicts several unit cells out of the infinite periodic medium considered. Part (**b**) shows the unit cell, which contains 40 pores, where the periodic boundary conditions are enforced. Part (**c**) shows detail of the finite-element discretization of the unit cell

imately isotropic. The calculations presented here correspond to the average[4] of 3 realizations in which the unit cell contains 40 pores.[5] Figure 4 shows one such realization. It is also appropriate to mention that, much like the two previous analytical results (2) and (12), the FE results presented here are based on a solution $\mathbf{F}(\mathbf{X})$ of equilibrium which does not necessarily correspond to the global energy-minimizing solution.

Figure 5(a) compares the hydrostatic response of the material with monodisperse pores described above with that of a HCA with the same matrix phase (17) and same initial porosity f_0. The results are shown in terms of the overall nominal stress $\overline{t}/\mu = 1/(2\mu)\mathrm{d}\overline{\phi}/\mathrm{d}\overline{\lambda}$ as a function of the applied deformation $\overline{\lambda}$ for $\alpha = 1$ (i.e., for Neo-Hookean matrix phase) and various values of f_0. Interestingly, the qualitative behavior of the material with monodisperse pores—in which the deformation gradient field in the pores is *not uniform*—is seen to be identical to that of the two-population-of-pores material shown in Fig. 3(b), namely: (i) it is stiffer than the HCA for large enough deformations $\overline{\lambda}$ and (ii) this difference in stiffness increases with decreasing values of initial porosity f_0. Accordingly, these results provide further evidence that the response (2) of the HCA is not an upper bound. And they also provide further indication that the hydrostatic stiffness of isotropic porous nonlinear elastic materials depends very strongly on the growth conditions of the underlying matrix phase, and not just on whether the deformation is radially symmetric.

The effect of the growth conditions of the matrix material is illustrated more directly in Fig. 5(b), where results are shown for the relatively small initial porosity $f_0 = 0.05$ for two values of α. The case of $\alpha = 1.2$ corresponds to a matrix phase with *stronger* growth conditions than the Neo-Hookean case ($\alpha = 1$) shown in Fig. 5(a), while the case of $\alpha = 0.7$ corresponds to a matrix phase with *weaker* growth conditions. The key observation from Fig. 5(b) is that matrix phases with stronger growth conditions lead to a behavior of the material with monodisperse pores that is increasingly stiffer than the behavior of the corresponding HCA. The trend of the results also suggests that the HCA response (2) may

[4]The maximum difference between results from different realizations is less than 1%.

[5]A parametric study of the problem with increasing number of pores in the unit cell indicates that 40 pores are sufficient for the FE model to be representative of an isotropic microstructure. The degree of isotropy was examined by checking the coaxiallity between the average Eulerian principal axes and the principal axes of the average Cauchy stress for various loadings.

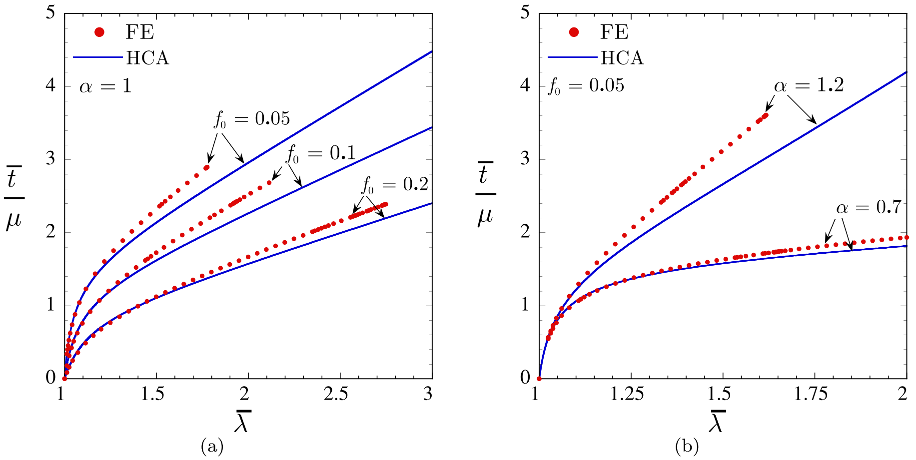

Fig. 5 Comparison of the hydrostatic response of a porous material with matrix phase (17) and random and isotropic distribution of monodisperse circular pores (*solid circles*) with the response of the corresponding HCA (*solid lines*). Results are shown in terms of the overall nominal stress $\overline{t}/\mu = 1/(2\mu)\mathrm{d}\overline{\phi}/\mathrm{d}\overline{\lambda}$ as a function of the applied hydrostatic deformation $\overline{\lambda}$ for various values of the initial porosity f_0 and the matrix material parameter α

remain an upper bound for the response of porous materials with matrix phases that have very weak growth conditions.

5 Final Remarks

In summary, the results put forward above have demonstrated that the hydrostatic response of a HCA, as defined by the effective stored-energy function (2), does *not* bound from above the hydrostatic response of isotropic porous *nonlinear elastic* solids. They have also demonstrated that radially symmetric deformation—contrary to common belief—is *not* necessarily the stiffest possible deformation mechanism to accommodate hydrostatic loading, but that other types of deformations may lead to stiffer responses depending on whether the growth conditions of the underlying matrix material are sufficiently strong. Finally, we remark that it would be interesting to extend the above results to 3D. It would also be interesting to carry out the corresponding analysis in the context of viscoplasticity, where there is much interest in clarifying whether the HCA is an extremal microstructure because of its intimate relation with the widely utilized Gurson's yielding criterion [24] for plastic and viscoplastic porous materials [25, 26].

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References

1. Hashin, Z.: The elastic moduli of heterogeneous materials. J. Appl. Mech. **29**, 143–150 (1962)
2. Hashin, Z., Shtrikman, S.: A variational approach to the theory of the elastic behaviour of multiphase materials. J. Mech. Phys. Solids **11**, 127–140 (1963)

3. Idiart, M.I.: Nonlinear sequential laminates reproducing hollow sphere assemblages. C. R., Méc. **335**, 363–368 (2007)
4. Lopez-Pamies, O., Idiart, M.I.: An exact result for the macroscopic response of porous Neo-Hookean solids. J. Elast. **95**, 99–105 (2009)
5. Perrin, G., Leblond, J.B.: Analytical study of a hollow sphere made of plastic porous material and subjected to hydrostatic tension—Application to some problems in ductile fracture. Int. J. Plast. **6**, 677–699 (1990)
6. Chaboche, J.-L., Suquet, P., Beson, J.: Endommagement et changement d'échelle. In: Bornert, M., Bretheau, T., Gilormini, P. (eds.) Homogénéisation en mécanique des matériaux, vol. 2, pp. 91–146. Hermes Science (2001)
7. Gărăjeu, M., Suquet, P.: On the influence of local fluctuations in volume fraction of constituents on the effective properties of nonlinear composites. Application to porous materials. J. Mech. Phys. Solids **55**, 842–878 (2007)
8. Moraleda, J., Segurado, J., Llorca, J.: Finite deformation of porous elastomers: a computational micromechanics approach. Philos. Mag. **87**, 5607–5627 (2007)
9. Hill, R.: On constitutive macrovariables for heterogeneous solids at finite strain. Proc. R. Soc. Lond. A **326**, 131–147 (1972)
10. Lopez-Pamies, O., Ponte Castañeda, P.: Second-order estimates for the macroscopic response and loss of ellipticity in porous rubbers at large deformations. J. Elast. **76**, 247–287 (2004)
11. deBotton, G.: Transversely isotropic sequentially laminated composites in finite elasticity. J. Mech. Phys. Solids **53**, 1334–1361 (2005)
12. Lopez-Pamies, O., Idiart, M.I.: Fiber-reinforced hyperelastic solids: A realizable homogenization constitutive theory. J. Eng. Math. **68**, 57–83 (2010)
13. Michel, J.C., Lopez-Pamies, O., Ponte Castañeda, P.: Triantafyllidis: Microscopic and macroscopic instabilities in finitely strained porous elastomers. J. Mech. Phys. Solids **55**, 900–938 (2007)
14. Hashin, Z.: Large isotropic elastic deformation of composites and porous media. Int. J. Solids Struct. **21**, 711–720 (1985)
15. Hashin, Z.: On elastic behaviour of fibre reinforced materials of arbitrary transverse phase geometry. J. Mech. Phys. Solids **13**, 119–134 (1965)
16. Sivaloganathan, J., Spector, S.J.: On the symmetry of energy minimizing deformations in nonlinear elasticity I: incompressible materials. Arch. Ration. Mech. Anal. **196**, 363–394 (2010)
17. Sivaloganathan, J., Spector, S.J.: On the symmetry of energy minimizing deformations in nonlinear elasticity II: compressible materials. Arch. Ration. Mech. Anal. **196**, 395–431 (2010)
18. Lopez-Pamies, O.: An exact result for the macroscopic response of particle-reinforced Neo-Hookean solids. J. Appl. Mech. **77**, 021016 (2010)
19. Idiart, M.I.: Modeling the macroscopic behavior of two-phase nonlinear composites by infinite-rank laminates. J. Mech. Phys. Solids **56**, 2599–2617 (2008)
20. Stuart, C.A.: Radially symmetric cavitation for hyperelastic materials. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **2**, 33–66 (1985)
21. Ball, J.M.: Discontinuous equilibrium solutions and cavitation in nonlinear elasticity. Philos. Trans. R. Soc. Lond. A **306**, 557–611 (1982)
22. Segurado, J., Llorca, J.: A numerical approximation to the elastic properties of sphere-reinforced composites. J. Mech. Phys. Solids **50**, 2107–2121 (2002)
23. Lopez-Pamies, O.: A new I_1-based hyperelastic model for rubber elastic materials. C. R., Méc. **338**, 3–11 (2010)
24. Gurson, A.L.: Continuum theory of ductile rupture by void nucleation and growth: Part I–Yield criteria and flow rules for porous ductile media. J. Eng. Mater. Technol. **99**, 2–15 (1977)
25. Leblond, J.B., Perrin, G., Suquet, P.: Exact results and approximate models for porous viscoplastic solids. Int. J. Plast. **10**, 213–235 (1994)
26. Danas, K., Idiart, M.I., Ponte Castañeda, P.: A homogenization-based constitutive model for isotropic viscoplastic porous media. Int. J. Solids Struct. **45**, 3392–3409 (2008)